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06

DESIGN OF RAINFALL NETWORKS USING ENTROPY

by

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ABSTRACT

Treating rainfall as a random field, models are derived using entropy for design of rainfall networks in space and time separately and jointly. The model for design in space offers an alternative to the model of Rodriguez-Iturbe and Mejia (1974). The model for design in time is extended using entropy spectrum for reconstruction of historical rainfall data. Long-term annual rainfall data for existing networks in Louisiana are used to verify the models. The efficiency of rainfall networks is evaluated using reduction of variance of mean areal rainfall. A comparison with some of the existing models shows that the proposed models are suitable, especially for data-scarce regions.

1. INTRODUCTION

Rainfall of an area is usually measured by a network of raingages. This network should be designed such that space-time variability of rainfall is sampled optimally. The network design has two modes: (1) number of raingages and positions for their installment (space design); and (2) time interval for measurement (or sampling interval) for each raingage and duration of the measurement program (time design). The information in one mode may be supplemented by the other with appropriate transfer mechanisms and by cross-correlation structure (space-time trade-off). Space-time design of rainfall networks should not be considered as a finished product, but should be updated periodically.

A design of rainfall network should consider the nature of the rainfall process to be sampled, and worth-of rainfall data to be obtained. The latter aspect has been helpful in historical development of network design. The worth is difficult to measure and is usually described by some other surrogates. An overview of studies on this aspect is given by Dawdy (1979). Langbein (1954) was probably the first to suggest the need to measure the worth of data. He proposed four ways for network design: (1) Use of statistical analysis and accuracy criteria as a measure of information. (2) Space-time trade-off. (3) Sample error as a major source of error in network design. (4) The influence of time dependency on the length of rainfall record.

Two approaches, employed for rainfall network design, have been: (1) information-variance, and (2) transfer function variance. Both approaches aim to satisfy the standard error or mean squared error (mse) criterion. For purposes of illustration, consider an observed rainfall time series to be described by some statistical parameters such as means, variances, correlations and cross-correlations. However, it is not known in advance which parameters are known (first error) and what their values (second error) are. Since the observed time series is of finite length, only estimates of the parameter values can be obtained. The difference between the mean of that estimate and its true value is the error which is usually measured by variance or second moment of the estimated parameter value. The latter is also known as the mean squared error (mse) and is defined as:

$$\text{mse} = E[x_{0t} - \sum_i w_i \, x_{it}]^2 \tag{1.1}$$

where x_{0t} is the unknown rainfall record at location 0 and time t and is estimated by weighting all known rainfall records at locations i each with some factor w_i. "E" denotes the expectation.

Information-variance may involve space-time trade-off explicitly. The decision whether to continue or discontinue a raingage is based on the relative gain of information. A network is designed by performing space and time analyses with some common measure of information. This idea, originated by Langbein (1954), was further extended by Rodriguez-Iturbe and Mejia (1974), and Lenton and Rodriguez-Iturbe (1977) using mathematical programming techniques. Another measure of information is cross-correlation transfer because rainfall series at different sites may be cross-correlated. Information about the rainfall amount at one site is transferable to another site (Fiering, 1965; Thomas and Benson, 1970). This has led to the concept of regional rainfall which has been used to evaluate the need for additional data collection both at gaged and ungaged sites (Matalas and Gilroy, 1968). The economic worth of data, together with rainfall information, has been used for design of economically efficient networks (Dawdy, et al., 1970; Moss, 1970; Moss and Dawdy, 1971).

Investigations using transfer function-variance employ spectral density function (spf). This function measures the distribution of the variance of rainfall occurrences over the range of frequencies inherent in their occurrences. A higher spf produces greater contribution to the variance. Eagleson (1967) used the spatial spf (transform of correlation-distance function) to determine optimal distances among raingages. An advantage of using transfer-function variance is that it reduces all the information concerning space-time dependence to a functional form. Some studies have been directed at the economic worth of data (Klemes, 1977; Slack, et al., 1975), decision theory and Bayesian analysis in rainfall network design (Duckstein, et al., 1974).

The objective of this study is to develop an approach based on entropy for space-time design of rainfall networks. Both variance and entropy are used as measures of information. Economic worth of data and Bayesian analysis are not considered. Designs in space and time are treated first separately, and then with the space-time trade-off.

2. MEASURES OF INFORMATION

2.1 VARIANCE AS A MEASURE OF NETWORK EFFICIENCY

The idea here is to position raingages at specified locations in space and use their measurements to estimate the mean rainfall depth over the area. The squared error of this estimate is the variance which decreases with increasing number of raingages. For a fixed raingage in time, variance measures the deviation of rainfall depth from time-averaged rainfall depth at that station, and decreases with increasing length of record.

Let x_{it} be rainfall depth at raingage i for time t, N total number of raingages and T length of record. Then, the space-time mean rainfall depth can be defined as:

$$\bar{x} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} x_{it} \tag{2.1}$$

for all raingages in the study area and for all available records. The space-time variance of the rainfall record can be defined as:

$$s_x^2 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it} - \bar{x})^2 \tag{2.2}$$

Assuming weak stationarity for the rainfall process, the mean rainfall depth $\bar{R}$ can be obtained from (2.1). Then, the variance of the mean rainfall $\bar{R}$ as a measure of the network efficiency is:

$$\text{var}(\bar{R}) = \text{var}[\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} x_{it}] \tag{2.3}$$

2.2 ENTROPY AS A MEASURE OF INFORMATION

Entropy is defined as expectation of information (Shannon, 1948). If X is a random variable and X_1, X_2, ..., X_n (with n = number of observations) are possible realizations of that random variable with occurrence probability p_k, k = 1, 2, ..., n, then the entropy H of X is:

$$H(X) = - \sum_{k=1}^{n} p_k \log(p_k) \tag{2.4}$$

Entropy measures the uncertainty associated with realization of X. If the realization of X is certain, then one of the probabilities p_k will be one and all others will be zero, and the entropy will then be zero as its minimum. With rising uncertainty about the realizations of X, entropy rises and achieves its maximum at log n. If entropy is maximized subject to the constraints based on prior knowledge, then the probability distribution corresponding to these will be least biased and consistent with respect to the constraints (Jaynes, 1978). This principle of maximum entropy (POME) has been successfully used in many scientific fields including hydrology (Sonuga, 1972, 1976; Amorocho and Espildora, 1973; Harmancioglu, 1980; Singh and Krstanovic, 1985; Singh, et al., 1985, 1986), but does not appear to have been applied to rainfall network design.

If X_1 and X_2 are two random variables whose joint probability of occurrence is $p_{i,j}$, $i = 1, \ldots, n_1$, $i = 1, \ldots, n_2$, then the joint entropy of X_1 and X_2 is:

$$H(X_1,X_2) = - \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} p_{i,j} \log(p_{i,j}) \tag{2.5}$$

Similarly, the conditional entropy of X_1, given X_2, is:

$$H(X_1|X_2) = - \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} p_{i,j} \log(p_{i|j}) \tag{2.6}$$

where $p_{i|j}$ is conditional probability of X_1, given X_2. Both joint and conditional entropies are similarly defined for m different random variables:

$$H(X_1, \ldots, X_m) = - \sum_{i} \ldots \sum_{m} p_{i,j,\ldots,m} \log(p_{i,j,\ldots,m}) \tag{2.7}$$

$$H(X_1|X_2, \ldots, X_m) = - \sum_{i} \ldots \sum_{m} p_{i,\ldots,m} \log (p_{i|j,\ldots,m}) \tag{2.8}$$

where $p_{i,j,\ldots,m}$ and $p_{i|j,\ldots,m}$ are respectively joint probability of $X_1, X_2, \ldots, X_m$ and conditional probability of X_1, given $X_2, \ldots, X_m$.

If $X_1, X_2, \ldots, X_m$, are stochastically independent random variables, then their joint entropy is the sum of marginal entropies:

$$H(X_1, X_2, \ldots, X_m) = \sum_{i=1}^{m} H(X_i) \tag{2.9}$$

If there exists dependence among the variables then the joint entropy is equal to the sum of the marginal entropy of one of the variables and conditional entropies of other variables as:

$$H(X_1, X_2, \ldots, X_m) = H(X_1) + \sum_{i=2}^{m} H(X_i | X_1, \ldots, X_{m-1}) \qquad (2.10)$$

Equation (2.10) gives the distribution of uncertainties among variables: the first term on the right side of the equation represents the uncertainty associated with realization of the variable X_1, the next term represents reduction in uncertainty of realization of X_1 by knowledge of X_2, X_3, etc. Joint and conditional entropies have been used by Harmancioglu (1980) in examining water pollution data.

3. DESIGN CONSIDERATIONS

3.1 RAINFALL DATA

There are nearly 100 raingages currently operating in the State of Louisiana, of which 76 were chosen with records longer than 15 years. Each raingage is located in one of the nine subregions of Louisiana which were demarcated by the U.S. National Weather Service in the 1950's. Each subregion supposedly represents the area of similar climatological characteristics. The abbreviations used to represent the subregions are: NW (northwest), NC (north central), NE (northeast), WC (west central), C (central), EC (east central), SW (southwest), SC (south central), SE (southeast). The raingages used are given in Table 1. Annual rainfall depth observed by each gage was used for analysis.

Table 1. The raingages used in the study.

Climatic Zones (Subregions)	Raingages
Northwest	Cotton Valley, Hosston, Minden, Plain Dealing, Rodessa, Shreveport WB Airport, Springhill
North Central	Calhan Exp. Station, Homer Exp. Station, Monroe FAA Airport, Ruston LA Polytech. Inst., Spearsville Fire Tower, Sterlington Look, Winfield 2W, Winona Fire Tower
Northeast	Bastrop, Epps 6W, Lake Providence, Saint Joseph Exp. Station, Winsboro
West Central	Ashland 2S, Converse, Grand Ecore, Hodges Gardens, Leesville, Many, Natchitoches
Central	Alexandria, Belah Fire Tower, Burkie, Grant Coteau, Marksville, Melville, Old River Lock, Opelousas, Simmesport, Vidalia No. 2
East Central	Amite, Baton Rouge WB Airport, Bogalusa, Covington 4NNW, Franklinton 3SW, Greenvell Springs, Hammond 3NW, Pearl River Lock No. 1, Pine Grove Fire Tower, Oaknolia, Sheridan Fire Tower, Slidell, Springville Fire Tower
Southwest	Crowley Exp. Station, DeQuincy 4N, DeRidder, Elizabeth, Hackberry 8SSW, Jennings, Kinder 3W, Lake Charles WB Airport, Longville, Mermentau, Oakdale, Oberlin Fire Tower, Rockfeller W1 Refuge, Vermilion Lock
South Central	Camille 2SW, Franklin 3NW, Jeanerette Exp. Station, Lafayette FAA Airport, Morgan City, New Iberia 5NW
South East	Donaldsonville 3E, Houma 1SW, New Orleans Algiers, Paradis 7S, Reserve, Saint Bernard.

3.2 MODES OF DESIGN

Three modes of design are considered: (1) Design in space, determining the sufficiency of the existing raingages and sampling distance. (2) Design in time, determining sampling time interval by examining dependence structure of raingage records and the spectral density function. (3) Design in space and time determining space-time trade-off by examining reduction of space-time variance and distribution of uncertainties among raingages by means of their marginal and conditional entropies. This procedure leads to an alternative to the space-time design model of Rodriguez-Iturbe and Mejia (1974).

The assumptions made for all three modes are: (a) the rainfall process is random in time and space; (b) long-term space-time mean of the rainfall depth is constant; and (c) variance is separable in both space and time.

3.3 SPACE-TIME VARIANCE

The variance of the mean rainfall from (2.3) is

$$\mathrm{var}(\bar{R}) = E[\bar{R} - E(\bar{R})]^2 = E[\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (x_{it}) - E(\bar{R})]^2 \quad (3.1)$$

This variance can be expressed as the product of the space-time variance of the rainfall record, and the temporal and spatial reduction factors F(T) and G(N) dependent only on sampling in time and space respectively (Rodriguez-Iturbe and Mejia, 1974) as:

$$\mathrm{var}(\bar{R}) = \sigma_x^2 \cdot F(T) \cdot G(N) \quad (3.2)$$

Consequently, sampling in space and in time can be treated separately. The factor G(N) is derived by Rodriguez-Iturbe and Mejia (1974) as:

$$G(N) = (\frac{1}{N^2}) \{N + N(N-1)\ E[r(d)]\} \quad (3.3)$$

where d is the distance between raingages and r(d) is the spatial correlation function. The form of F(T) depends on the order of autocorrelation function.

3.4 DETERMINATION OF CORRELATION FUNCTION

To determine correlation between raingages in Louisiana, one central raingage was chosen in each subregion, and correlations between the residuals of rainfall depths of that raingage and every other raingage were determined in terms of lag-zero cross-correlation coefficient. Let the subscripts i and c denote an i-th raingage and the central raingage respectively. The variable x_{it} denotes the rainfall depth of i-th raingage and time t, and T represents the length of record of the raingage i. Then, lag-zero cross-correlation coefficient is:

$$r_{ci}(0) = \frac{\sum_{t=1}^{T} (x_{ct} - \bar{x}_{ct})(x_{it} - \bar{x}_{it})}{[\sum_{t=1}^{T} (x_{ct} - \bar{x}_{ct})^2 \sum_{t=1}^{T} (x_{it} - \bar{x}_{it})^2]^{1/2}} \qquad (3.4)$$

Values of cross-correlations are plotted as isocorrelation lines around each central raingage as shown in figure 1. The subregions NW, NC, NE were highly correlated since each was inside 0.90 isocorrelation line. The subregions with relatively weak correlation structure were SW, C and SC (the boundary was 0.6 isocorrelation line). A wider range of correlations was obtained when the whole of Louisiana was considered as one area. Correlations of selected raingages which are farthest apart and all other (75) raingages are shown as points in figure 2.

4. DESIGN IN SPACE

4.1 DERIVATION OF SPATIAL REDUCTION FACTOR

It is assumed that raingages can be located randomly. Let d be the distance between raingages, $\bar{d}$ mean distance, and σ_d^2 variance of distances. The probability density function (pdf) of distances, derived using POME as a normal distribution in Appendix A, can be rewritten as:

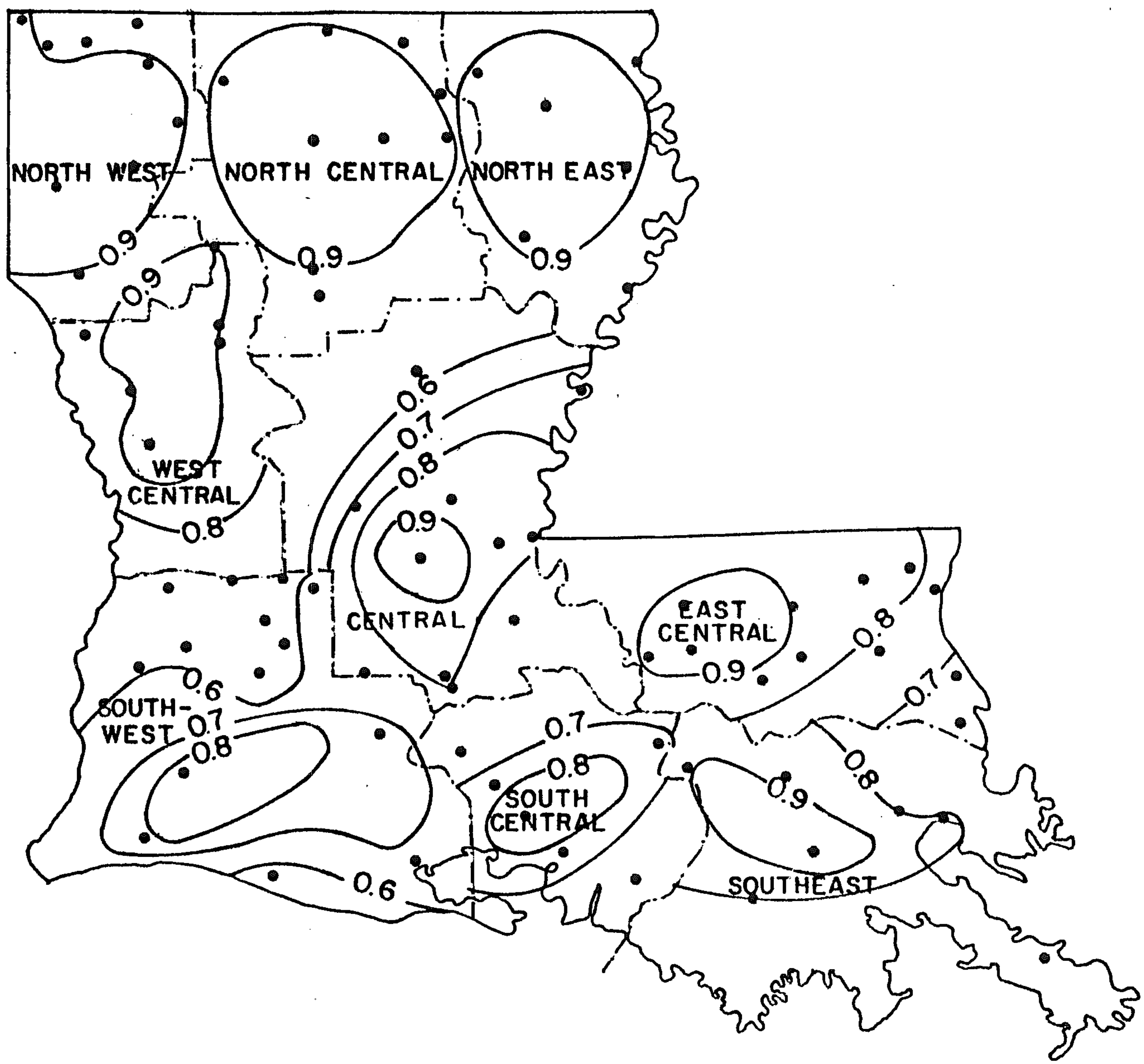

Figure 1. Isocorrelation lines for Louisiana subregions for 15 years of record.

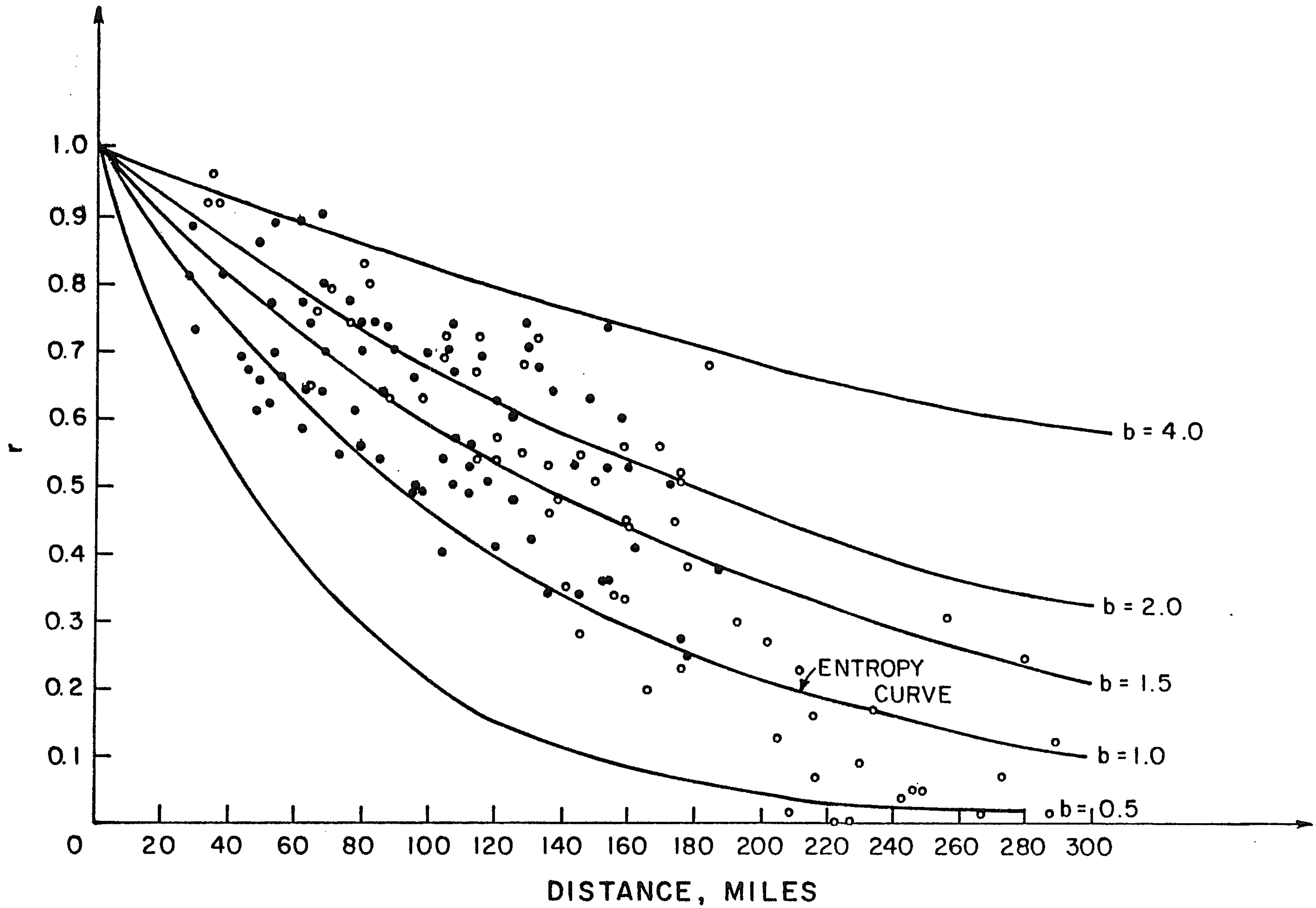

Figure 2. Cross-correlation function of residuals of rainfall depths for distances between different raingages.

$$p(d) = \frac{1}{[2\pi(\sigma_d^2)]^{0.5}} \exp[-\frac{(d-\bar{d})^2}{2\sigma_d^2}] \qquad (4.1)$$

In other words, spatial sampling follows a normal probability distribution.

The spatial correlation function r(d), treated as a pdf derived using POME as an exponential distribution in Appendix B, can be written as:

$$r(d) = \frac{1}{\bar{d}} \exp(- d/\bar{d}) \qquad (4.2)$$

This form must be modified by the factor of proportionality $k = \bar{d}$, since at zero distance $r(d=0) = 1$,

$$r(d) = \exp(- d/\bar{d}) \qquad (4.3)$$

A plot of correlation function r(d) for some selected Louisiana raingages is shown in figure 2. For d between 220 and 300 miles, r(d) became negligible. Equation (4.3) is a special case of:

$$r(d) = \exp(- d/b\bar{d}) \qquad (4.4)$$

where b is dispersion factor. From figure 2, b appeared to be in the range (1,1.5). For $b > 1$, r(d) diminished with d at a much slower rate. For Louisiana, $b = 1$ (obtained using entropy) was more realistic.

The mean of the spatial correlation for a given area is:

$$E[r(d)|A] = \int_0^{r_{max}} r(d)\ g(r(d))\ dr \qquad (4.5)$$

where r_{max} is the maximum spatial correlation for area A. Rodriguez-Iturbe and Mejia (1974) expressed (4.5) as

$$E[r(d)|A] = \int_0^{r_{max}} r(d)\ f(d)\ dd \qquad (4.6)$$

where d_{max} is the maximum distance among two raingages in area A and f(d) is the frequency function of the distance d between any two randomly chosen raingages. The frequency function can be expressed by p(d) given in (4.1), and r(d) given in (4.4). Since the left side of (4.6) does not include area explicitly, one can write:

$$E[r(d)] = \int_0^{d_{max}} \frac{1}{[2\pi(\sigma_d^2)]^{0.5}} \exp[- \frac{d}{\bar{d}} - \frac{(d-\bar{d})^2}{2\sigma_d^2}]\, dd \qquad (4.7)$$

Using the notations $\lambda_1 = 1/\bar{d}$ and $\lambda_2 = 1/2\sigma_d^2$, and integrating on the whole probability space $(-\infty,+\infty)$:

$$E[r(d)] = \frac{1}{[2\pi(\sigma_d^2)]^{0.5}} \int_{-\infty}^{+\infty} \exp[- \lambda_1 d - \lambda_2 (d-\bar{d})^2]\, dd \qquad (4.8)$$

Using the formula (3.322) given in Gradshtein and Rhyzhik (1980), (4.8) can be solved as:

$$E[r(d)] = \exp\{[- \frac{\bar{d}^2}{2\sigma_d^2}] + \frac{\sigma_d^2}{2} (\frac{1}{\bar{d}^2} + \frac{\bar{d}^2}{\sigma_d^4} - \frac{2}{\sigma_d^2})\}$$

$$\cdot\ [1 - \phi(\frac{\sigma_d}{2^{0.5}\ \bar{d}} - \frac{\bar{d}}{2^{0.5}\ \sigma_d})]$$

where $\phi(.)$ is the error function. A simplification of this equation expresses mean of the spatial correlation only as a function of c_v or coefficient of variation ($c_v = \sigma_d/\bar{d}$) as:

$$E[r(d)] = \exp(\frac{c_v^2}{2} - 1) \cdot [1 - \phi(\frac{c_v^2-1}{2^{0.5}\ c_v})] \qquad (4.9)$$

By inserting (3.9) in (2.3), the spatial reduction factor is obtained as:

$$G(N) = \frac{1}{N^2} \{N + N(N-1) \exp(\frac{c_v^2}{2} - 1) [1 - \phi(\frac{c_v^2-1}{2^{0.5}\ c_v}]\} \qquad (4.10)$$

This gives the reduction in variance due to sampling in space. Equation (4.10) was plotted for various values of c_v while keeping N constant. For $c_v > 5.6$, G(N) became constant regardless of N as shown in figures 3

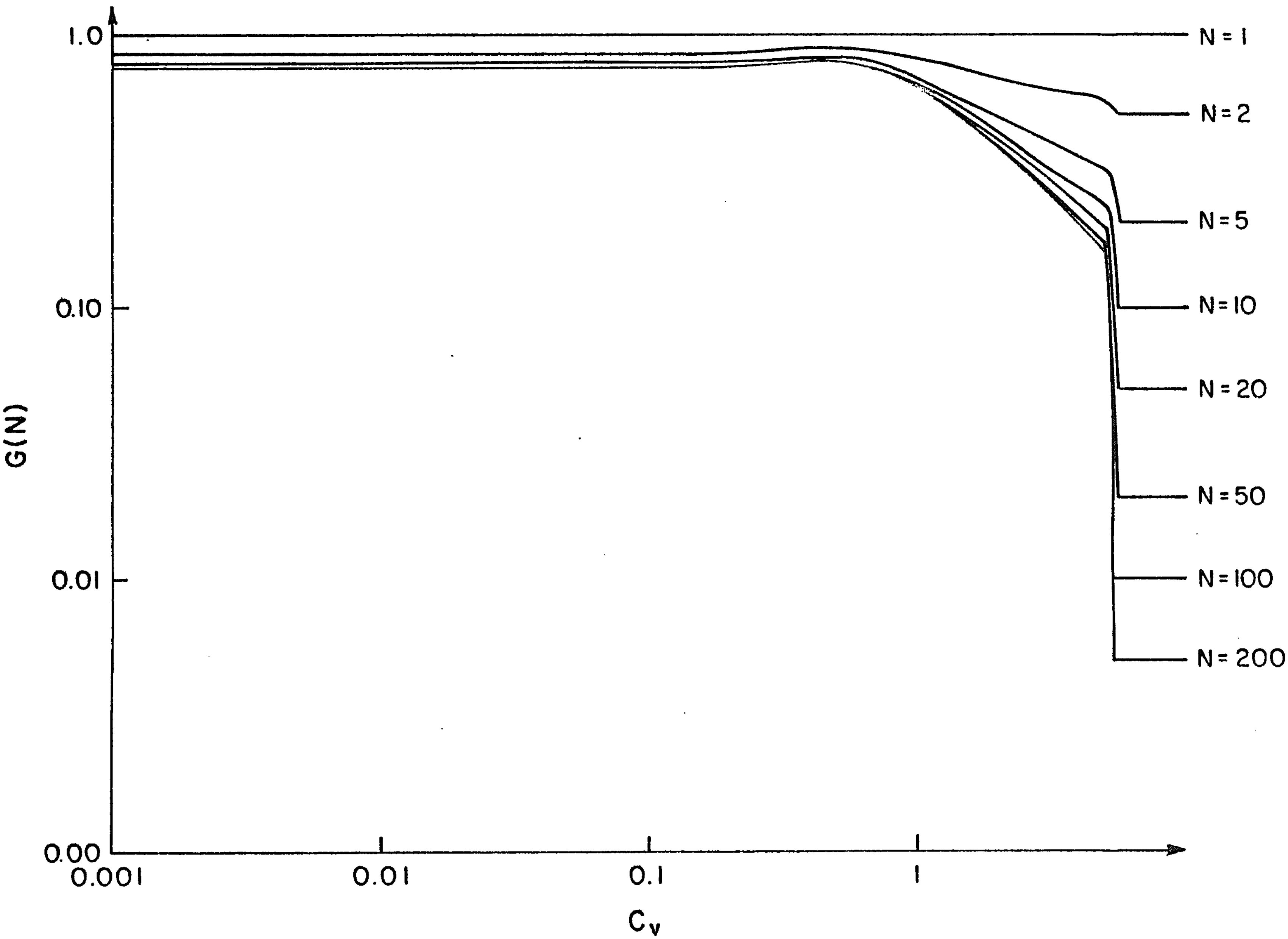

Figure 3. Variance reduction factor due to spatial sampling for various values of the coefficient of variation and number of raingages (broad range).

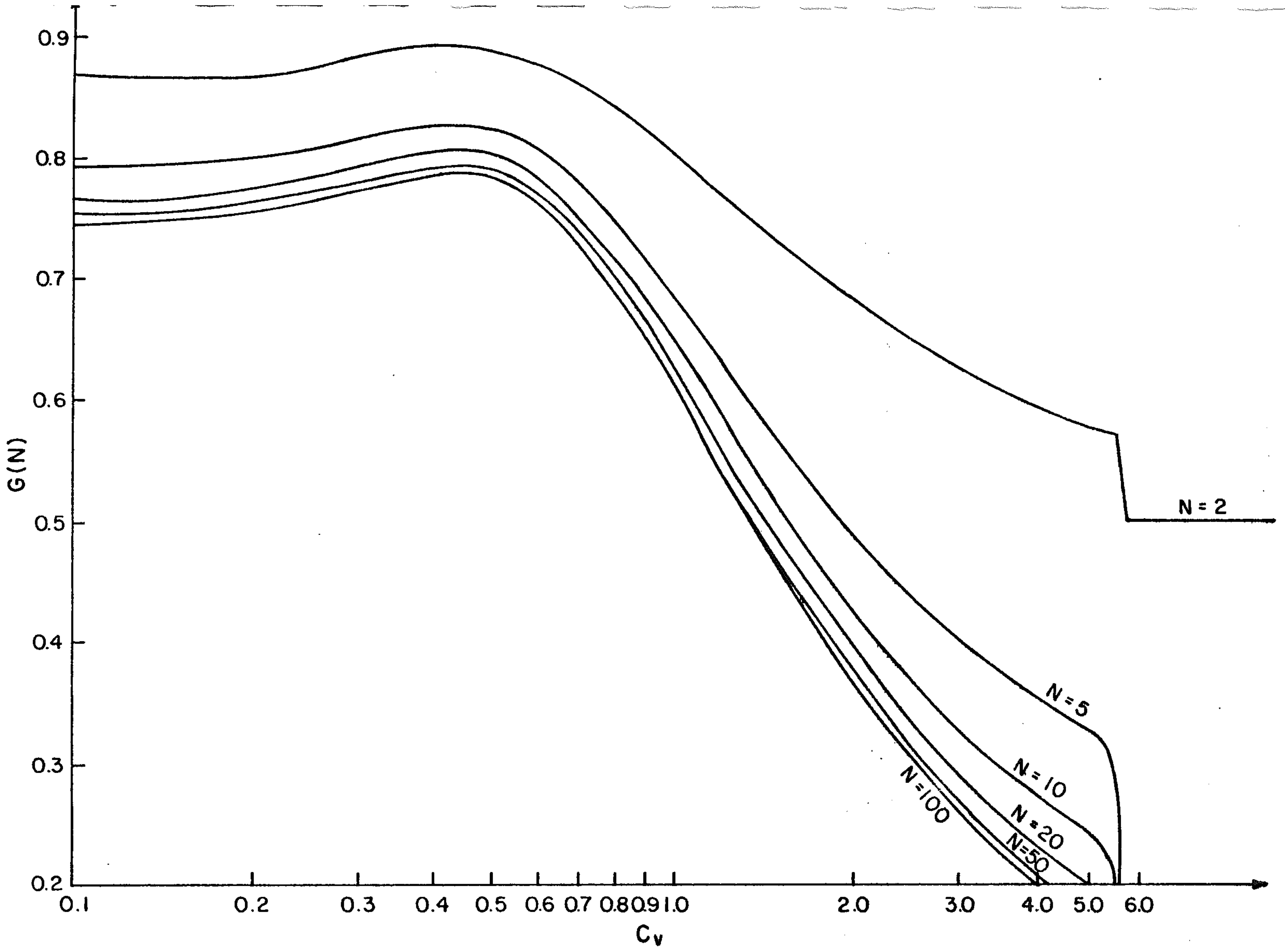

Figure 4. Variance reduction factor due to spatial sampling for various values of the coefficient of variation and number of raingages (narrow range).

and 4. The realistic range of c_v is (0.1, 5.6), covering virtually all meaningful values of means and variances of distances. Some values of G(N) for various values of c_v and N are given in Table 2.

Table 2. Spatial reduction factor for various values of the coefficient of variation (c_v) and the number of stations (N).

N / c_v	9	18	27	36	45
0.5	0.805	0.79	0.785	0.73	0.71
0.6	0.785	0.77	0.76	0.758	0.756
0.7	0.75	0.74	0.73	0.725	0.72
0.8	0.72	0.70	0.695	0.69	0.688
0.9	0.685	0.668	0.656	0.654	0.650
1.0	0.655	0.630	0.620	0.615	0.611

4.2 COMPUTATION OF VARIANCE REDUCTION

For known distances among raingages, sample estimates of $\bar{d}$, σ_d^2, and c_v were calculated. Then, G(N) was obtained by decreasing or increasing the number of raingages. When c_v was greater, which was for greater variance of the distances, a greater range of reduction was obtained. For a greater combination of possible distances, e.g., for Louisiana $\{{76 \atop 2}\}$, a simplified procedure was applied to expedite calculations. This procedure is a consequence of simple random sampling (Cochran, 1977). For nine subregions of Louisiana, one raingage was selected in each subregion; then all possible distances (9 x 75) with respect to these nine points were measured. For every sample of distances, its $\bar{d}$ and σ_d^2 were estimated. The sample mean of all possible distances is an unbiased estimate of the real mean:

$$\hat{\bar{d}} = \frac{1}{9} \sum_{i=1}^{9} \hat{\bar{d}}_i \tag{4.11}$$

where $\hat{\bar{d}}$ is the estimate of $\bar{d}$ for the entire State of Louisiana and $\hat{\bar{d}}_i$ is estimate of $\bar{d}$ for the subregion i. Similarly, the variance for the entire State of Louisiana S_d^2 was computed as

$$S_d^2 = \frac{1}{9} \sum_{i=1}^{9} S_d^2(i)$$

in which $S_d^2(i)$ is the variance estimated for the subregion i. S_d^2 is an estimate of σ_d^2. This variance should be corrected by finite population correction (9/76) as:

$$S_d^2 = \hat{S}_d^2(1 - 9/76) \tag{4.12}$$

The values of mean and variance for Louisiana were found to be: $\bar{d}$ = 127.50 miles; S_d^2 = 4,362 miles^2; and $c_v \cong 0.52$.

For any subregion with 9 raingages, the value of N can be determined from figures 3 and 4. For example, for central subregion, $\bar{d}$ = 41 miles, S_d = 17.8 miles, and c_v = 0.43, the variance reduction factor was G(N) = 0.81.

4.3 COMPUTATION OF ENTROPY

Entropy was computed for the random sampling of distances in Louisiana, considering one or more raingages per subregion. The entropy of (4.1) is equivalent to the one of the normal distribution function (Singh, et al., 1985):

$$H(X=d) = \frac{1}{2} \ln [\ S_d^2\ (2\pi e)] \tag{4.13}$$

The entropy H(X=d) measures the uncertainty of sampling distances among the available raingages. Nine Louisiana raingages or one raingage in each subregion have the entropy $H(d_1)$. When the number of raingages is increased, the sampling variation grows. For 2 raingages in each subregion, we have entropy $H(d_1,d_2)$. The remaining uncertainty is $H(d_1|d_2)$ and is computed from (2.10). The uncertainty is decreased by further increasing the number of raingages. The remaining uncertainty is always equal to the difference among the successive joint entropies. The results are presented in Table 3. The most significant entropy reduction occurred when the number of raingages increased from one to two in each subregion or from 9 to 18 for the entire State of Louisiana.

Table 3. Entropy (napiers) matrix for space design with the distance among the raingages as the random variable.

Number of Stations per Subregion	Total Number of Stations	Distance Statistics		Entropy H(d) [napiers]	Conditional Entropy [napiers]
		$\bar{d}$ [miles]	S_d [miles]		
1	9	130	53.44	5.40	5.40
2	18	136	56.88	5.47	0.07
3	27	147	62.00	5.54	0.07
4	36	150	66.00	5.60	0.06

5. DESIGN IN TIME

5.1 CORRELATION STRUCTURE OF RAINFALL TIME SERIES

The autocovariance functions c_k, derived in Appendix C, for the time series of rainfall depth can be written as:

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp(ik\theta)\,d\theta}{\sum_{j=0}^{m} \lambda_j \exp(ij\theta)} \tag{5.1}$$

where m is the number of constraints and λ_k the Lagrange multipliers of the entropy method. In (0,m) interval, values of c_k are determined from known data; for $k > m$, c_k is determined from (4.1). By using all c_k's (from 0 to the last known lag of the historical record), one may reconstruct original record of the rainfall depths.

It is important to enumerate the significance of the Lagrange multipliers λ_k. In Appendix C, it is proved that each new λ_k is equal to the inverse of the autocovariance at that lag:

$$\lambda_k = 1/c_k \tag{5.2}$$

Its role is analogous to the role of partial autocorrelation function ϕ_{kk} in time series analysis. While ϕ_{kk} denotes correlation introduced at lag k not accounted for by previous k-1 lags, λ_k denotes information introduced at lag k not accounted for by k-1 previous lags. For example, elements of the Toeplitz matrix in (C-5) are explained as: λ_0 on the main diagonal represents information introduced by the variance, λ_1 represents information introduced by autocovariance of lag one, and λ_m represents information introduced by autocovariance of lag m. All other λ_k's $(k > m)$ are redundant and do not bring new knowledge.

5.2 TESTING DEPENDENCY OF RAINFALL RECORDS

The annual rainfall time series of each raingage was tested for independence using autocorrelation function expressed as:

$$r_k = \frac{\sum_{t=1}^{T-k} (x_t - \bar{x})(x_{t+k} - \bar{x})}{\sum_{t=1}^{T} (x_t - \bar{x})^2} \tag{5.3}$$

A plot of r_k for some selected raingages is shown in figure 5. The 95% confidence limits, given by Anderson (1941), are very far indicating that the rainfall series is highly independent.

5.3 EVALUATION OF DEPENDENCY ORDER

The Lagrange multipliers λ_k were calculated to see how much information was introduced at different lags. Except for λ_0, of particular interest are λ_1 and λ_2, representing lag-one and lag-two dependency, respectively. For the 9 Louisiana subregions, all new λ_k's were calculated for autocorrelation of areal mean values and plotted against the lag k as shown in figure 6. This illustration is only approximate, since in theoretical derivation of (C-4), the time base is assumed infinitely long, whereas the data is available for only 15 years. The λ_1 was dominant for 6 regions of higher correlations and introduction of new λ_k's produced redundant knowledge. For the regions of weak correlations (SC, C, SW), λ_2 was more important. To get valid approximation for more lags, every raingage should be considered separately and its records extended by double-mass curve or some other techniques. This was done for one of the raingages (figure 7) where the records were extended 35 years in the past giving the total time base of 50 years. New calculations again showed strong dominance of λ_1.

5.4 EVALUATION OF TEMPORAL VARIANCE REDUCTION FACTOR

On the average, use of first-order dependency was considered sufficient for all Louisiana raingages. This points out adequacy of using c_k or r_k of lag one. Using r_k, the variance reduction factor F(T) was derived by Rodriguez-Iturbe and Mejia (1974) as:

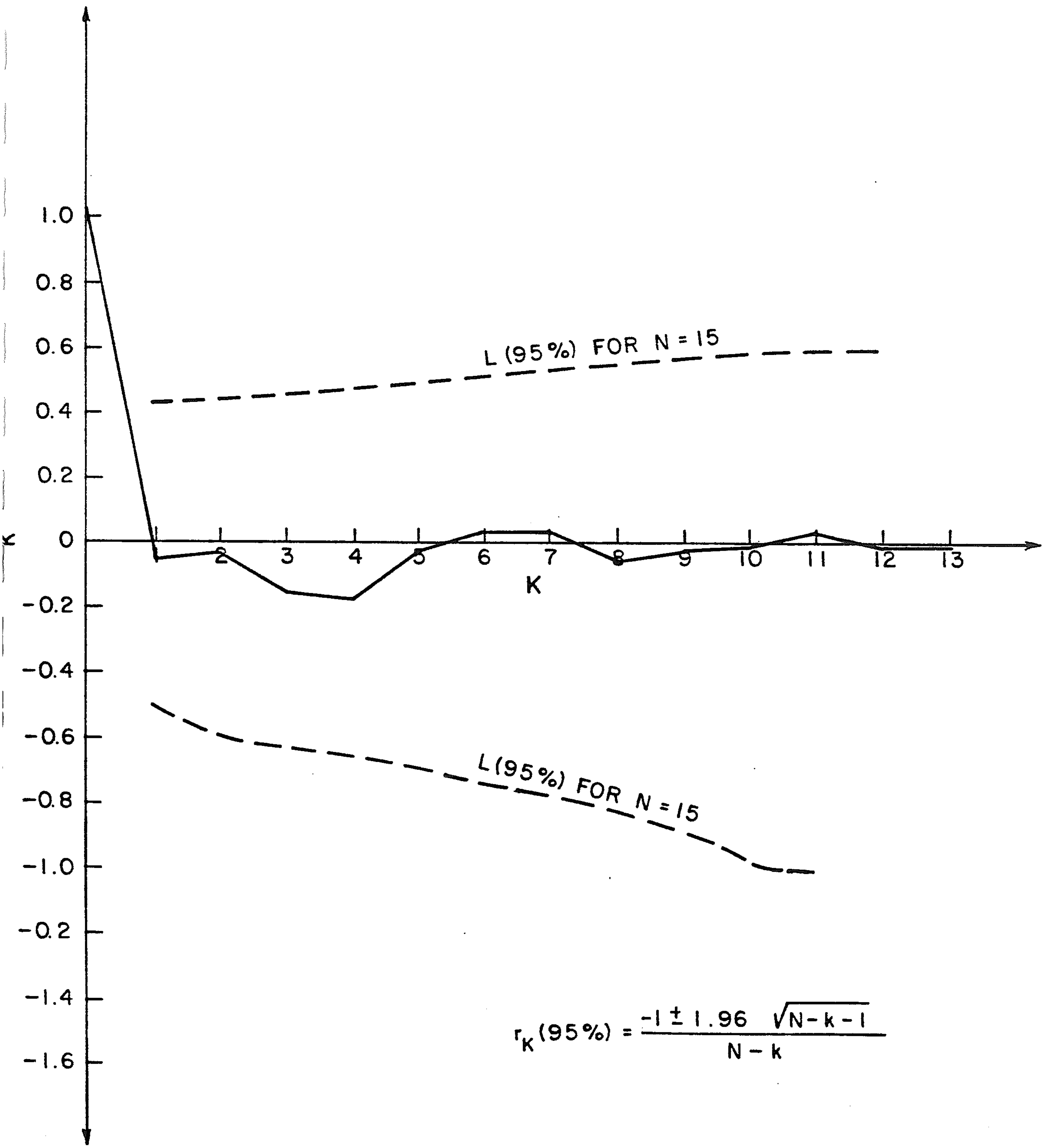

Figure 5. Correlogram of annual rainfall for selected raingages.

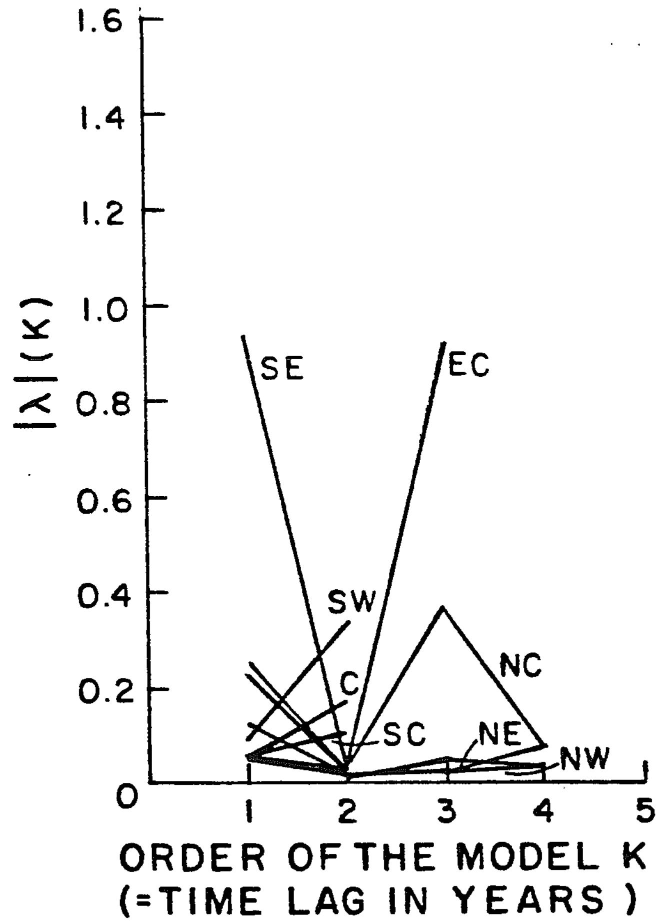

Figure 6. Relation between the Lagrange multipliers of the entropy model and the order of the time series model for the nine subregions in Louisiana.

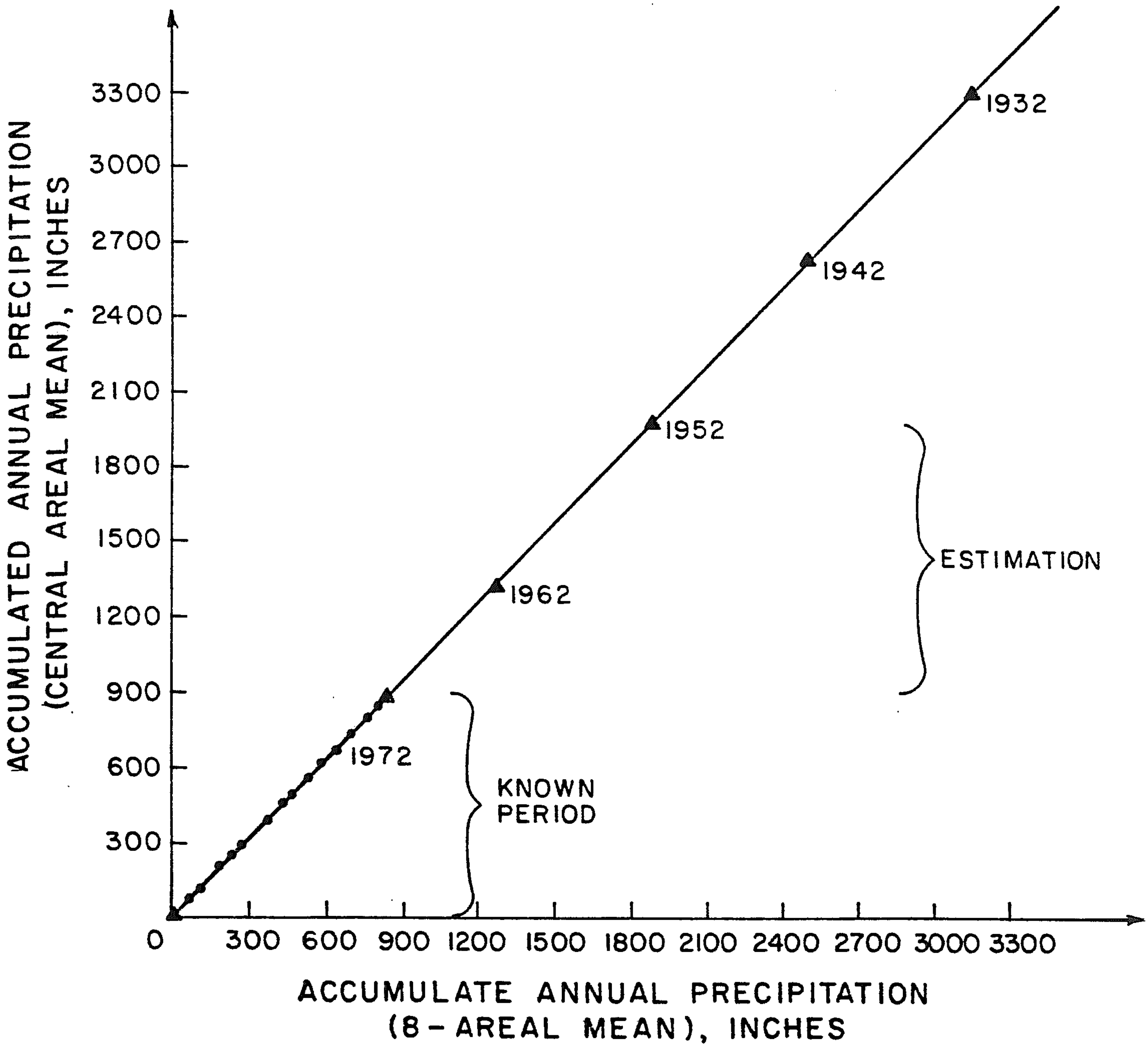

Figure 7. Double mass curve for extending the annual rainfall records by 35 years in the past.

$$F(T) = \frac{1}{T^2} \{T + 2 \frac{r_1}{1 - r_1} [T - 1 - \frac{r_1}{1 - r_1} (1 - r_1^{T-1})]\} \tag{5.4}$$

F(T) was plotted for various values of r_1 and T as shown in figure 8 . The value of r_1 was obtained as 0, which was somewhat expected according to the very low value of r_1 from the correlogram of figure 5. This confirms that the rainfall in any year does not depend on the previous year rainfall.

5.5 CALCULATION OF ENTROPY SPECTRA

Having determined all the Lagrange multipliers for a certain number of lags (e.g., $k = 10 \rightarrow \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{10}$ = 11 multipliers), one can determine the power spectrum since it is related to c_k by

$$P(f) = \sum_{k=-\infty}^{+\infty} c_k \cos(2\pi fk) \tag{5.6}$$

for frequency $|f| \leq 0.50$, or by using (C-13), one can predict entropy spectrum P(f) as

$$P(f) = \frac{1}{\sum_{k=-m}^{m} \lambda_k \cos(2\pi fk)}, \qquad |f| \leq 0.5 \tag{5.7}$$

Equation (5.7) was originally derived by Burg (1975) and then Jaynes (1982). Because of the summetry of the autocovariance function, (C-13) can be written as:

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp(ik\theta)\,d\theta}{\sum_{k=-m}^{+m} \lambda_k \exp(ik\theta)}$$

or

$$c_k = \frac{1}{2\pi i} \int_{|z|=1} \frac{z^{k-1}\,dz}{\sum_{k=-m}^{+m} \lambda_k z^k}, \quad z = \exp(i\theta) \tag{5.8}$$

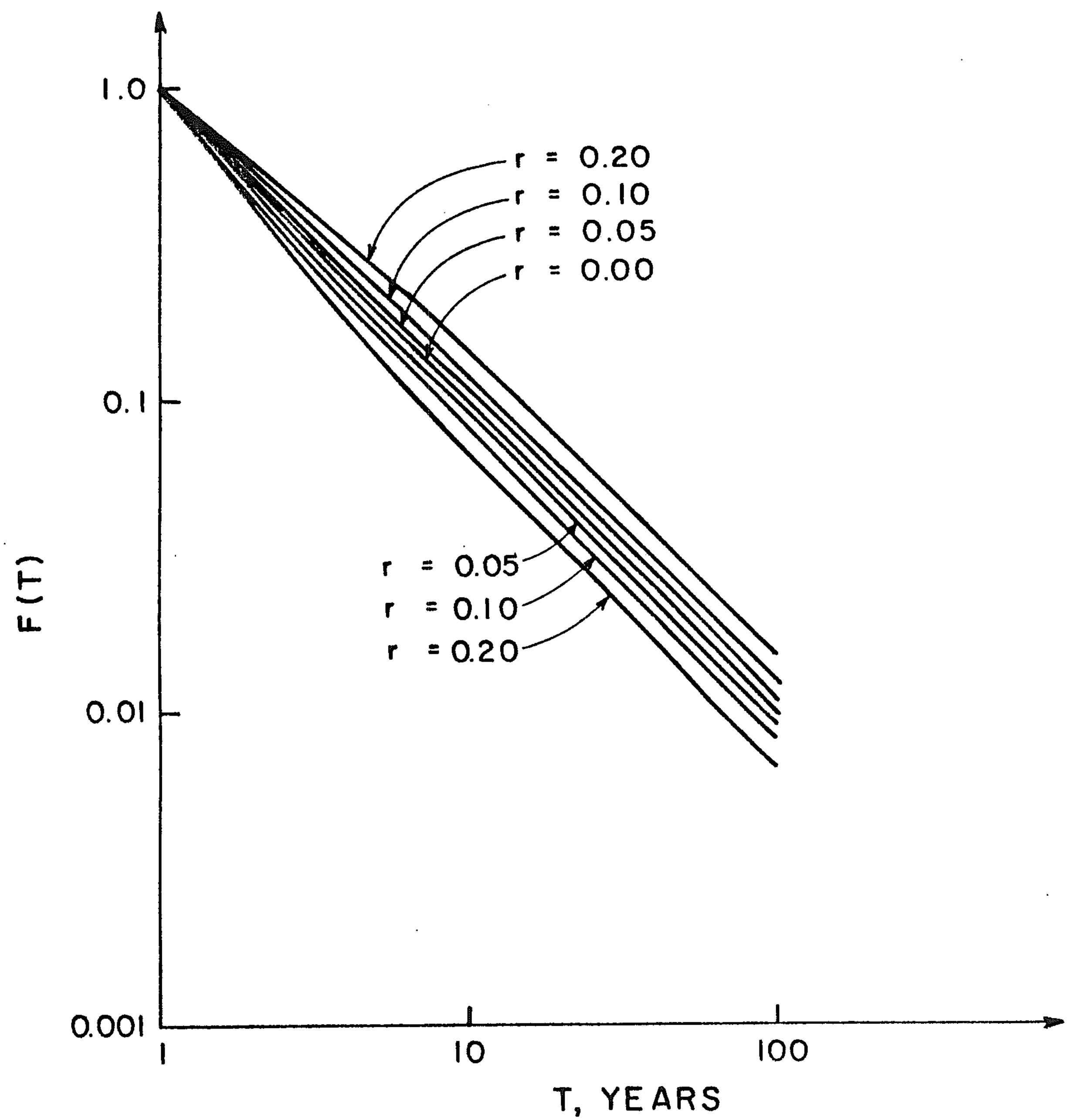

Figure 8. Variance reduction factor due to temporal sampling for various first order autocorrelation coefficients.

Denominator of (5.8) can be expressed as the product of two polynomials such that the roots of the first polynomial lie outside the unit circle and the roots of the second polynomial inside the unit circle,

$$\sum_{k=-m}^{+m} \lambda_k z^k = \frac{1}{P_m} \left(\sum_{k=0}^{m} a_k z^k \right) \left(\sum_{k=0}^{m} a_k^* z^{-k} \right) \qquad (5.9)$$

where P_m is constant >0, and a_k^*, a_k are coefficients. Note that $a_0 = 1$. In general, a_k^* is the complex conjugate of a_k. For the real values $a_k = a_k^*$, and one can write (5.7) as

$$P(f) = \frac{P_m}{\left| \sum_{k=0}^{m} a_k z^{-k} \right|^2} = \frac{P_m}{\left[\sum_{k=0}^{-m} a_k \cos(2\pi fk) \right]^2 + \left[\sum_{k=0}^{-m} a_k \sin(2\pi fk) \right]^2} \qquad (5.10)$$

The values of a_k were obtained first using the Levinson-Burg algorithm (Burg, 1975) presented in Appendix D. Then the entropy spectrum was determined from (5.10). This entropy spectrum was compared with unsmoothed power spectrum of historical records for each Louisiana subregion. The unsmoothed power spectrum was computed as:

$$P(f) = 1 + 2 \sum_{k=1}^{+m} c_k \cos(2\pi fk) \qquad (5.11)$$

As shown in figures 9-15, entropy spectra yielded reasonably accurate predictions, although their peaks were slightly higher than those for the original record.

The Lagrange multipliers were determined from (5.9) by equating the terms of equal powers as:

$$\lambda_0 = (1 + \sum_{k=1}^{m} a_i^2)/P_m$$

$$\lambda_j = \lambda_{-j} = (a_j + \sum_{k=1}^{m-j} a_k a_{k+1})/P_m \qquad (5.12)$$

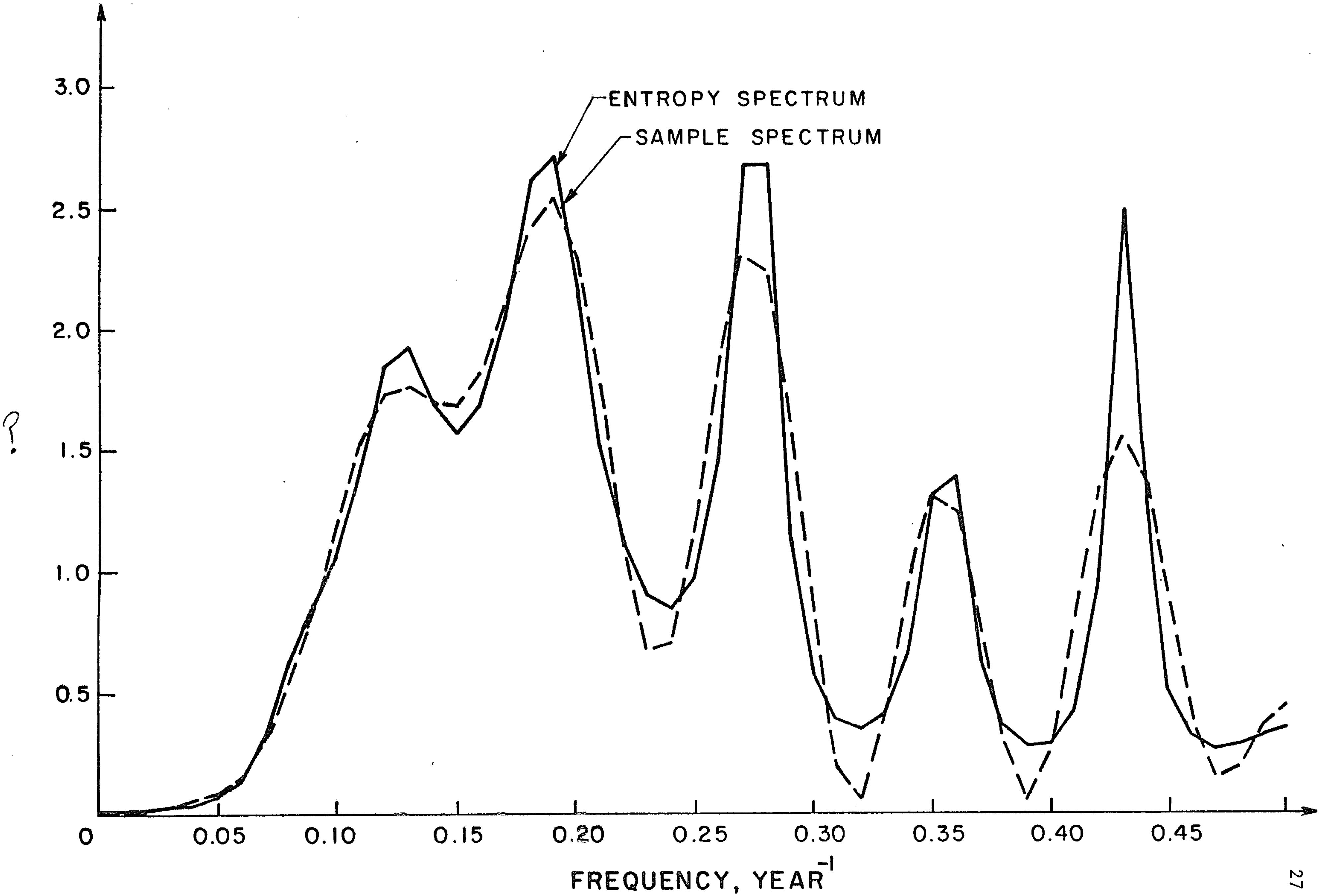

Figure 9. Comparison of the entropy spectrum and sample spectrum for the north central subregion.

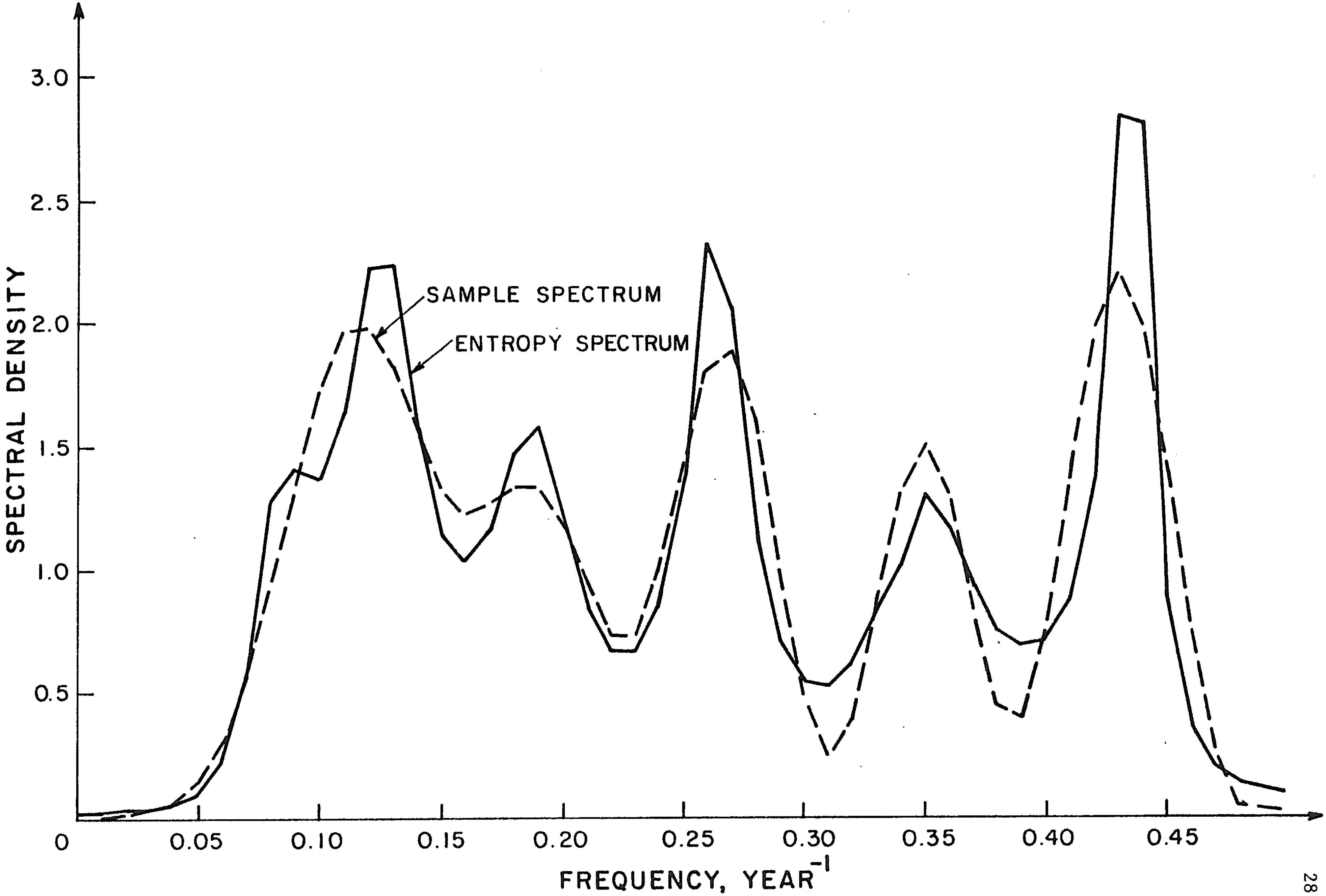

Figure 10. Comparison of the entropy spectrum and sample spectrum for the northeast subregion.

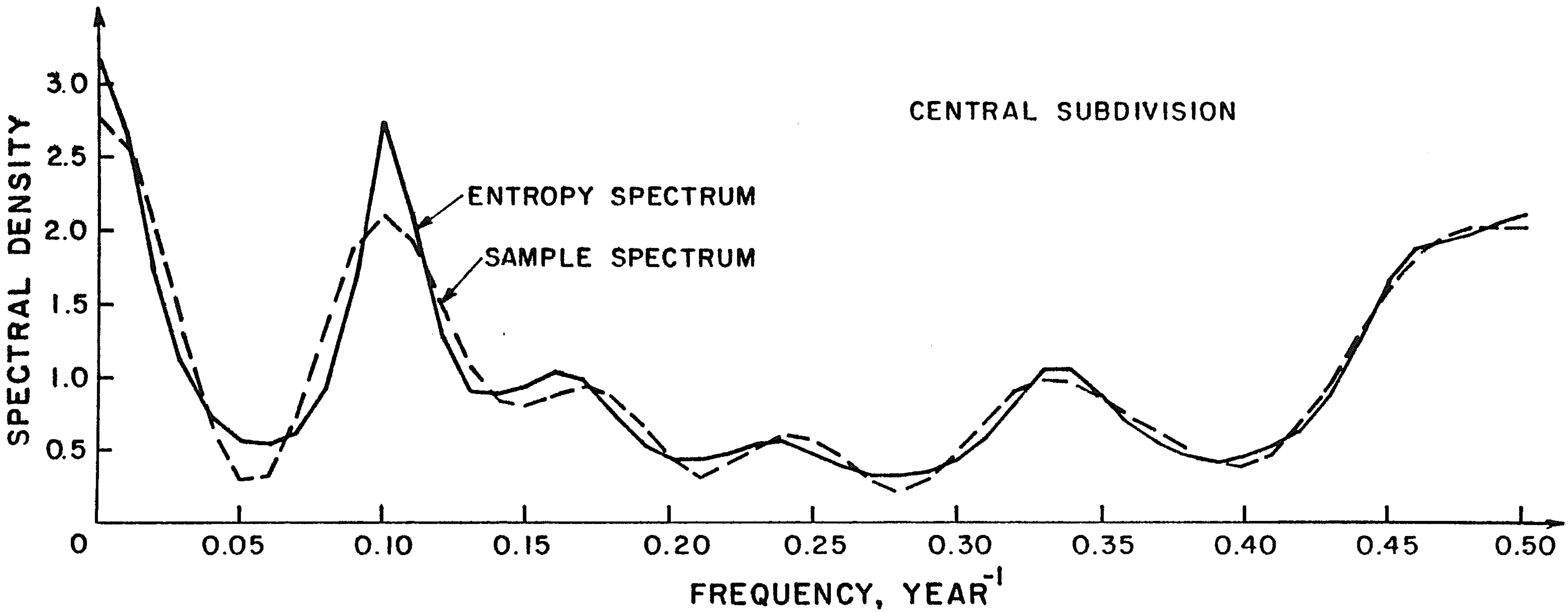

Figure 11. Comparison of the entropy spectrum and sample spectrum for the northwest subregion.

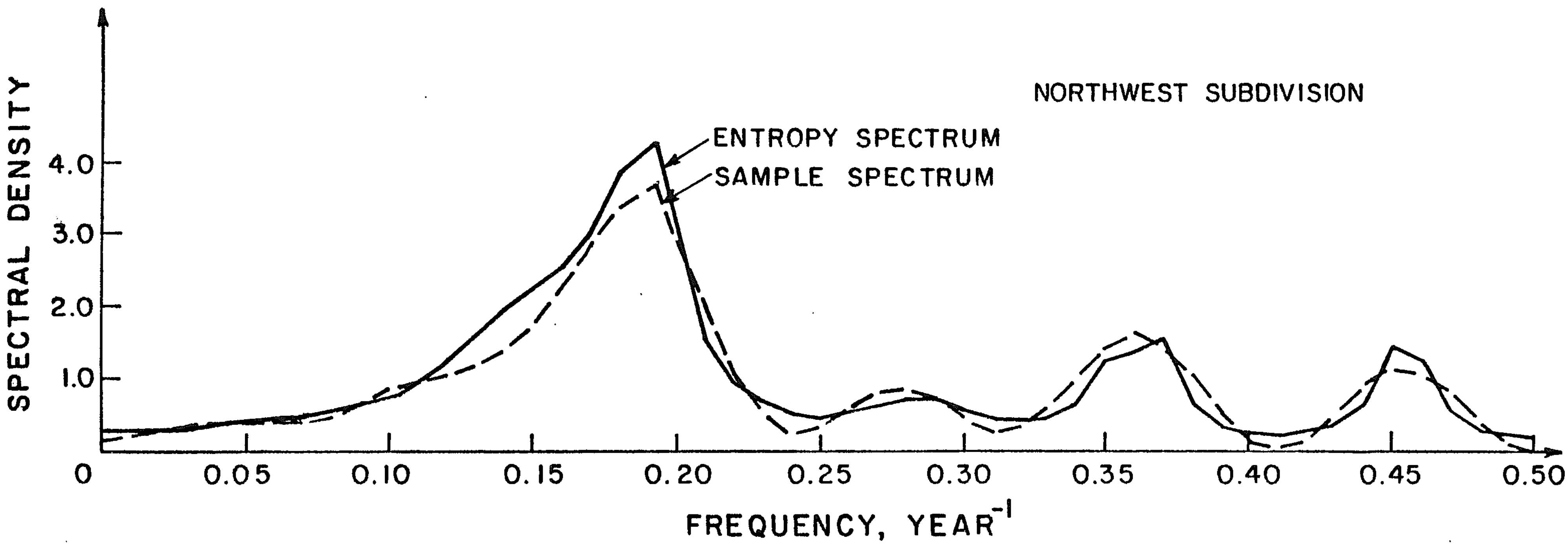

Figure 12. Comparison of the entropy spectrum and sample spectrum for the central subregion.

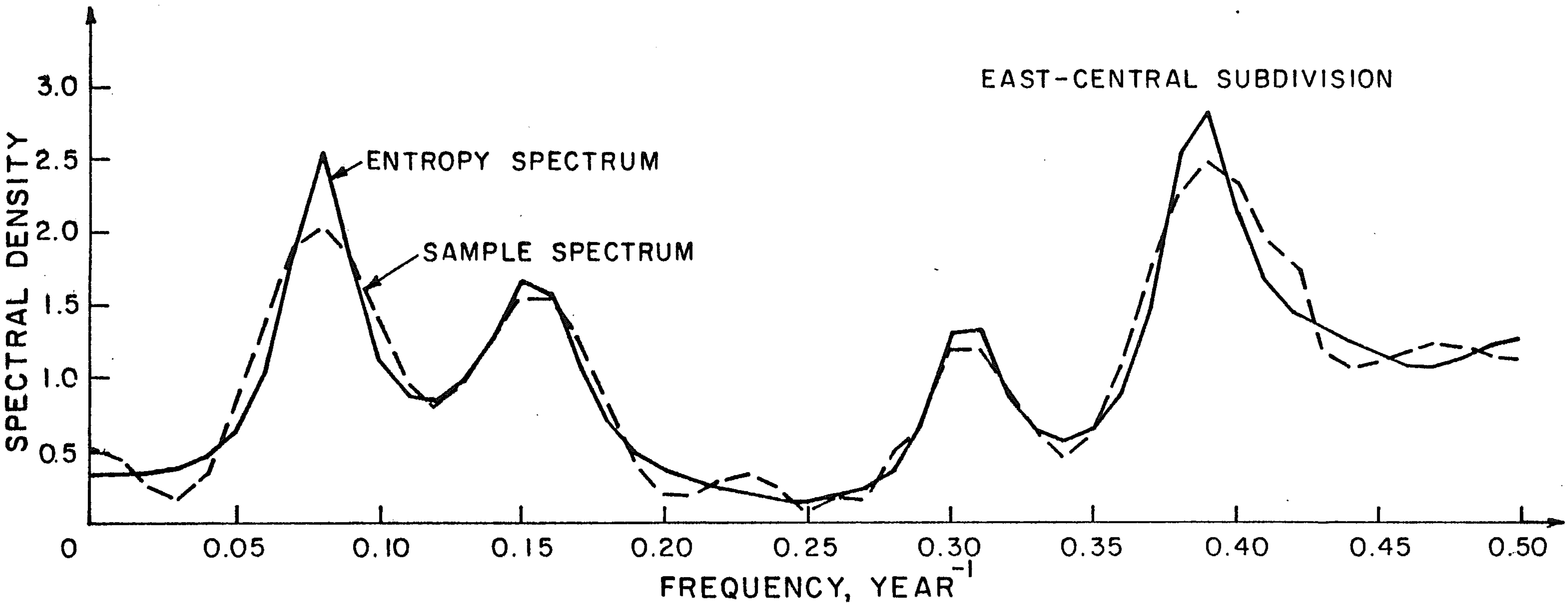

Figure 13. Comparison of the entropy spectrum and sample spectrum for the east central subregion.

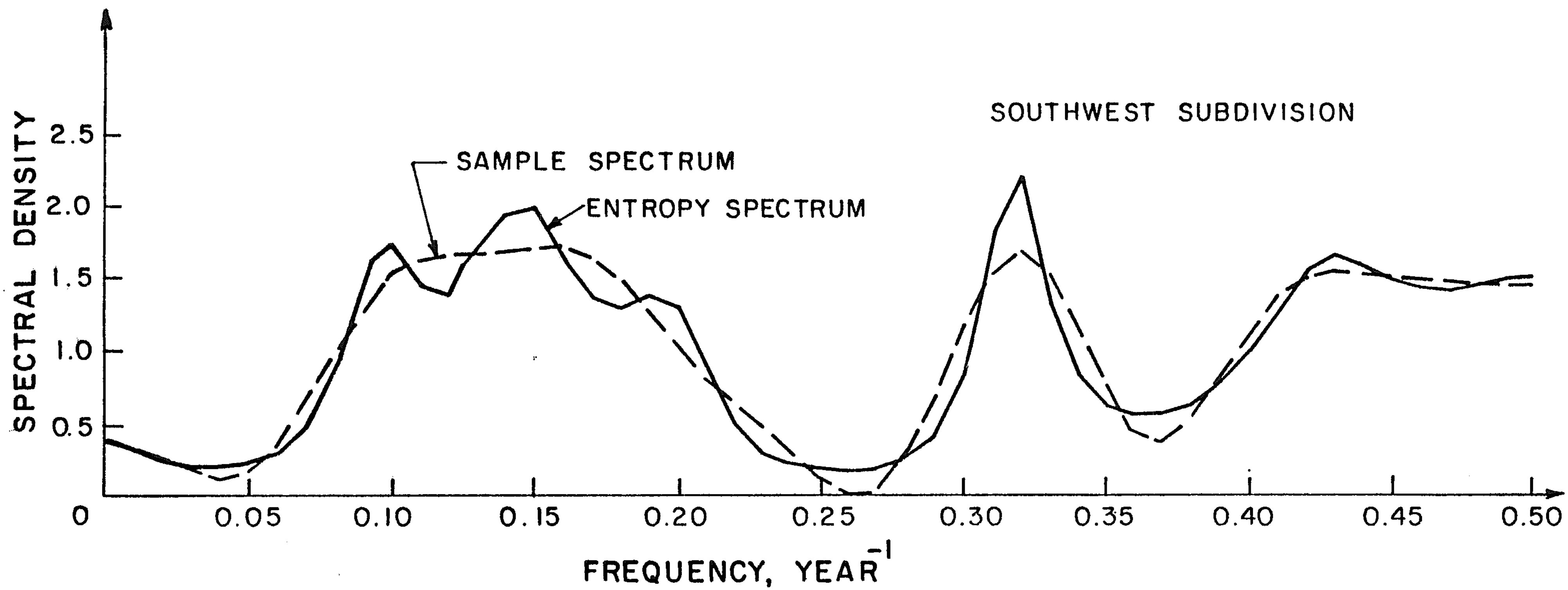

Figure 14. Comparison of the entropy spectrum and sample spectrum for the southwest subregion.

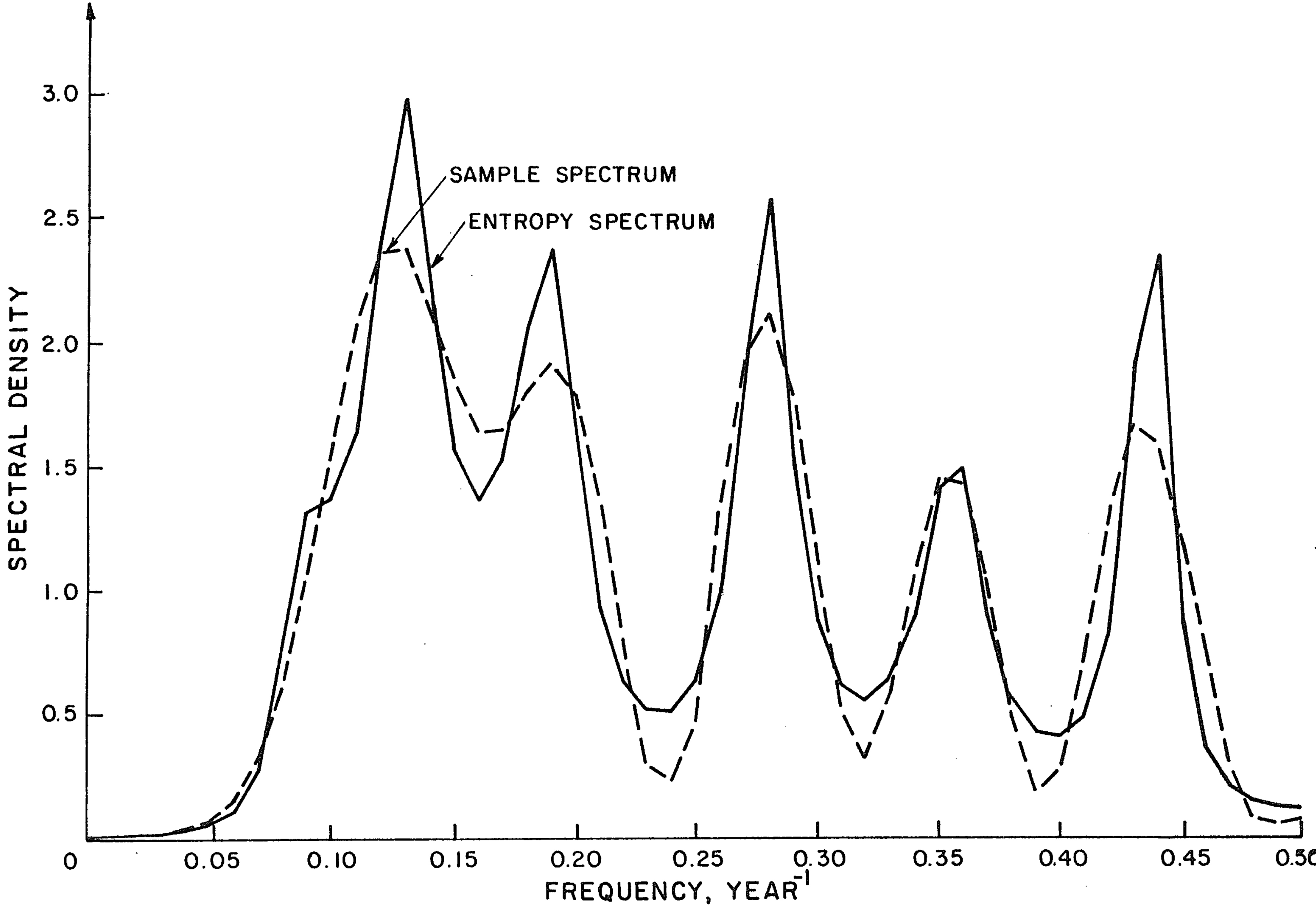

Figure 15. Comparison of the entropy spectrum and sample spectrum for the west central subregion.

$\lambda_{-m} = \lambda_m = a_m/P_m$

The value of λ_k, presented in Table 4, produced sharp spectral peaks only if its magnitude was much higher than its neighboring λ's: λ_{k-1} and λ_{k+1} (e.g., EC, NW, C or SW subdivision). If all high λ_k's were concentrated one after the other, it was difficult to determine the ones responsible for formation of spectral peaks (e.g., NE, WC or NC subregions).

These spectral peaks show relative contributions to the variance of mean rainfall. The NC and NE subregions have significant contributions at frequencies 0.18 and 0.26 which correspond to the period of 5.5 and 3.8 years, the subregion C has significant peak at 0.10 or every tenth year, and the southern subregions have significant contributions to the variance at both smaller and higher frequencies.

It is concluded that subregions of higher correlations have one or two frequencies or periods when significant rainfall depths occur. The subregions of weaker correlations have more equal distributions of rainfall depths for most frequencies or periods.

6. SPACE-TIME DESIGN

6.1 TOTAL REDUCTION IN VARIANCE

The total reduction in variance for both space and time was computed as $F_{total} = G(N) * F(T)$. The values of G(N) and F(T) were obtained as:

N (stations)	1	5	10	20	50	100
G(N)	1.0	0.82	0.79	0.785	0.780	0.777

T (years)	1	5	10	15	20	30	40	50	60	100
F(T)	1	0.2	0.10	0.067	0.05	0.033	0.025	.02	.013	.010

Table 4. Lagrangian multipliers for 15 lags obtained from (5.12).

Lag k	λ_K	Division						
		NW	NC	NE	WC	C	EC	SW
0	λ_0	1.87	6.30	4.83	7.96	1.41	1.82	1.82
1	λ_1	-.14	4.44	2.72	5.84	-0.06	0.22	0.37
2	λ_2	0.50	4.90	3.64	6.51	-0.33	-0.60	-.13
3	λ_3	0.28	4.40	2.56	5.41	0.13	0.06	0.049
4	λ_4	0.31	3.85	3.20	5.52	0.066	0.76	0.65
5	λ_5	0.06	3.34	1.95	4.23	0.086	0.25	0.13
6	λ_6	-0.10	3.00	2.47	4.02	-0.196	-0.28	-0.32
7	λ_7	0.07	2.27	1.29	2.74	0.020	-0.11	-0.01
8	λ_8	0.12	1.96	1.63	2.71	-.033	0.36	0.29
9	λ_9	-0.12	1.39	0.74	1.61	-0.230	0.24	-0.26
10	λ_{10}	0.26	1.22	1.06	1.53	0.04	-.22	-0.39
11	λ_{11}	-0.34	0.59	0.25	0.58	-0.03	-.05	-0.01
12	λ_{12}	0.32	0.50	0.44	0.72	-0.13	-0.105	-0.05
13	λ_{13}	-0.09	0.57	0.23	0.40	-0.03	-0.103	-0.18
14	λ_{14}	-.009	-.07	0.15	0.08	0.070	-0.096	-0.13
15	λ_{15}	.005	.23	-.04	0.62	0.076	0.031	-0.03

For various values of raingages N and years T, F_{total} is given in Table 5.

Table 5. Values of total reduction in variance for various values of N and T.

N \ T	2	5	10	15
1	0.50	0.200	0.100	0.067
2	0.44	0.176	0.088	0.0590
3	0.43	0.172	0.086	0.0576
5	0.41	0.164	0.082	0.0549
10	0.395	0.158	0.079	0.0529
20	0.3925	0.157	0.0785	0.0526
100	0.388	0.155	0.078	0.0521

Notations: N = number of raingages
T = number of years

The space-time mean rainfall depth was $\bar{\bar{R}}$ = 58.96 inches using (2.1), and space-time variance of the raingall record was $s_x^2 = 149.80$ inches^2 using (2.2). Finally, space-time variance of the mean rainfall was computed using (3.2):

$$\text{var}(\bar{\bar{R}}) = 149.80\ F_{total}$$

which gives:

$$\text{var}(\bar{\bar{R}}) = 8.1 \text{ inches}^2 \text{ for 1 raingage/subregion operating 15 years}$$

or

$$\text{var}(\bar{\bar{R}}) = 7.9 \text{ inches}^2 \text{ for 2 raingages/subregion operating 15 years}$$

By increasing the number of raingages, the variance reduction was effective until the density of 1 to 2 raingages per subregion. More dense raingages decreased the variance only marginally. This is evident in the last computation where the precision of only 0.2 inches^2 is achieved by doubling the number of raingages.

6.2 USE OF ENTROPY IN SPACE-TIME DESIGN

Entropy is used here to measure the distribution of uncertainties associated with measurement of rainfall in all 9 subregions. Let X be a vector of N variables (X_1, ..., X_N), with each X_i ($i = 1, \ldots, N$) represented by a time series with a time base of 15 years. Specifically, X_1 represents the time series of central raingage for NW subregion, X_2 for NC, X_3 for NE, X_4 for WC, X_5 for C, X_6 for EC, X_7 for SW, X_8 for SC and X_9 for SE. Using (C-4), entropy of the random vector X is:

$$H(X) = \frac{N}{2} \ln 2\pi - \frac{1}{2} \ln |\Lambda| + \frac{N}{2} \qquad (6.1)$$

where $|\Lambda|$ is determinant of the matrix of the Lagrange multipliers given by (C-5). Using similarity between multivariate normal distribution and (C-4), it is noted that $|\Lambda| = |\Sigma|^{-1}$, where $|\Sigma|$ is either autocovariance matrix (for $N = 1$ and the entropy $H(X_i)$ of the single variable is calculated) or cross-covariance matrix (when $N > 1$) with cross-correlation between related variables X_i inside the random vector X. For space design, (6.1) was used with N as the number of raingages in the same subregion. Computations, presented in Table 6, were performed as follows. First, one raingage in each subregion was chosen (usually central raingage) and its entropy $H(X_1)$ was computed. Then a new raingage of the same subregion was added and the joint entropy $H(X_1,X_2)$ was computed. The relationship between two raingages is given by their covariance structure represented by Λ matrix. The conditional entropy $H(X_1|X_2)$ represents the amount of uncertainty left in X_1 when X_2 was

Table 6. Entropy matrix for space design (entropy [napiers]) with the rainfall depth as a random variable.

SPACE DESIGN

Subregion	$H(X_1)$	$H(X_1 \mid X_2)$	$H(X_1 \mid X_2,X_3)$	$H(X_1 \mid X_2,X_3,X_4)$	$H(X_1 \mid X_2,\ldots,X_5)$
NW	3.938	3.035	2.592	2.592	2.459
NC	4.059	3.460	2.846	2.775	2.707
NE	4.092	3.216	3.183	3.166	2.904
WC	3.873	3.347	2.689	2.643	2.642
C	3.804	3.352	3.115	3.111	3.068
EC	3.608	3.155	2.969	2.934	2.899
SW	3.555	3.345	3.273	2.961	2.796
SC	3.723	3.265	3.169	3.065	2.922
SE	3.800	3.395	3.238	2.853	2.850

Notations: $H(X_i \mid X_j)$ = conditional entropy at raingage i given the raingage j in same subregion.

introduced. It was computed as a difference between the joint and marginal entropy according to (2.10). This procedure was repeated until 5 raingagesin each subregion were introduced. The most significant reduction in entropy occurred from one raingage to two raingages per subregion. Introduction of additional raingages produced negligible decrease in entropy.

For time design, (6.1) was used again with N = 1. X is now the rainfall record for selected raingages in each subregion. The results are presented in Table 7. Several cases were distinguished with sampling information every year, every second year, and every third year. Greater sampling intervals were not examined because of the short

Table 7. Entropy matrix for time design (entropy [napiers]).

TIME DESIGN

Subregion	Time Interval (years)	$H(X_1)$	$H(X_1 \| X_2)$	$H(X_1 \| X_2,X_3)$	$H(X_1 \| X_2,X_3,X_4)$	$H(X_1 \| X_2,..,X_5)$
NW	1	3.783	2.359	2.316	2.270	2.2353
	2	3.693	2.274	2.262		
	3	3.713				
NC	1	3.923	2.502	2.447	2.417	2.392
	2	3.876	2.405	2.340		
	3	3.804				
NE	1	3.899	2.480	2.466	2.453	2.422
	2	3.764	2.336	2.195		
	3	3.882				
C	1	3.704	2.258	2.250	2.233	2.214
	2	3.409	1.990	1.667		
	3	3.865				
EC	1	3.685	2.255	2.240	2.239	2.183
	2	3.509	2.090	1.823		
	3	3.631				
SW	1	3.604	2.176	2.1759	2.150	2.111
	2	3.303	1.876	1.488		
	3	3.815				

Table 7. (continued)

Subregion	Time Interval (years)	$H(X_1)$	$H(X_1 \mid X_2)$	$H(X_1 \mid X_2,X_3)$	$H(X_1 \mid X_2,X_3,X_4)$	$H(X_1 \mid X_2,\ldots,X_5)$
SC	1	3.538	2.091	2.060	2.030	2.011
	2	3.457	2.024	1.970		
	3	3.718				
SE	1	3.640	2.219	2.186	2.184	2.139
	2	3.723	2.266	2.206		
	3	3.206				

Notations: $H(X_1 \mid X_2,\ldots,X_j)$ ($j = 2,\ldots,5$) = entropy at a selected station given the time series rainfall record of lag j.

time record. For each case, conditional entropies were computed using (1.10). X_1 now represents the full time series, and X_k the series with lag k-1. Each conditional entropy $H(X_1|X_2, \ldots, X_k)$ represents the uncertainty remaining at a specific raingage when the information from k other lags is known. The relationship among the lags is expressed by the autocovariance matrix represented by Λ.

7. DISCUSSION

7.1 DISCUSSION OF RESULTS

From Table 5 it is seen that the influence of F(T) on the total reduction factor F_{total} was much greater than F*(N). The number of raingages needed for accurate measurement of long-term areal rainfall in Louisiana depends on the reduction in variance to be achieved. For example, for existing 76 raingages operating for 15 years, the reduction of 5.2% was achieved as shown in Table 5. That is hardly economical since 5.3% reduction in variance was already achieved by keeping 18 raingages or 2 in each subregion. By further increasing the number of raingages, one could decrease the reduction by 0.1%, which is hardly economical. Indeed, 5.5% precision was achieved with only one station per subregion.

The results of Table 3 show the applicability of the random sampling in the network design. The procedure depends only on the geometry of the area and its raingage configuration. The results of Table 6 depend on the rainfall data. Both tables clearly show sufficiency of two raingages per subregion. The reduction of uncertainties due to introduction of more raingages was insignificant. Table 7 shows that significant correlation at lag 1 existed, while dependence at higher lags was negligible: from lag 0 to lag 1, the reduction of uncertainty was maximum for all subregions. This confirmed the results of the analysis of the matrix of Lagrange multipliers: the dominance of λ_0 and λ_1 associated with lags 0 and 1. Additionally, the sampling in time every second year produced the uncertainties of the same order as for every year, thus two year records may be useful for longer time series.

7.2 COMPARISON OF RESULTS

The results of this study (the curves of figure 3) were compared with those of Rodriguez-Iturbe and Mejia (1974, figures 7). The parameter λ_3 was related to their correlation parameter h as:

$$\lambda_3 = A\ h^2$$

where A is the area of rectangle that approximates the shape of Louisiana ($A = 44{,}520$ miles^2). For the area of central Venezuela they obtained a reduction in variance by 2.5% for 20 stations operating for 15 years which is very close to the results of this study.

Another important comparison is with the work of Eagleson (1967). To this end, the area of Louisiana was approximated by a triangle and the mean distance between any two randomly chosen points was computed by comparing that triangle with the one of unit area and multiplying by the factor of 0.5544 (Mattern, 1960). The correlation distance obtained in this way was $r_0 = 17$ miles. Then the parameter β, as defined by Eagleson, is $\beta = A^{0.5}/r_0 = 1.80$. Using Eagleson's curves for the optimum number of raingages for water yield studies (Eagleson, 1967, figure 9), 9 raingages yielded 0.32 normalized variance of the sample mean (or 0.26 for 18 stations). For Louisiana, the normalized variance of space-time mean rainfall would be $var(\bar{R}) = 0.60$ that gives 4 raingages from Eagleson's curves sufficient for Louisiana. This is comparable with our results, since the space-time variance is not exactly equal to Eagleson's and his curves were also derived for much smaller areas.

8. CONCLUSIONS

1. For time design, the entropy method is proposed to determine the order of autocorrelation coefficient. The Lagrange multipliers are interpreted in their new role of explaining the entropy model.
2. The entropy spectrum is derived for reconstruction of historical records. It also explains the distribution of information per frequency or period different for the subregions of higher and lower correlation.
3. For space design, an alternative to the model of Rodriguez-Iturbe and Mejia is suggested.
4. Space-time trade-off is performed by computing the entropy matrix that gives the distribution of uncertainties per subregion. It is concluded that one year time interval sampling must be preserved even for the area greater than Louisiana.
5. One station per subregion is shown to be effective both for reduction of variance and computation of individual entropies. This is also validated by comparing results with the previous investigations.
6. The results of this study are confined to long-term areal rainfall only.

9. ACKNOWLEDGEMENTS

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10. REFERENCES

Amorocho, T. and Espildora, B., 1973. Entropy in the assessment of uncertainty in hydrologic systems and models. Water Resources Research, Vol. 9, No. 6, pp. 1511-1522.

Anderson, R. L., 1941. Distribution of the serial correlation coefficients. Annals of Mathematical Statistics, Vol. 8, No. 1, pp. 1-13.

Burg, J. P., 1975. Maximum entropy spectral analysis. Ph.D. thesis, Stanford University, University Microfilms, 75-25, 499.

Cohran, W. G., 1977. Sampling Techniques. John Wiley and Sons, 428 pp.

Davis, D. R., 1971. Decision making under uncertainty in hydrology. Technical Report 2, Hydrology and Water Resources Dept., University of Arizona, Tucson, Arizona.

Davis, D. R. and Dvoranchik, W. M., 1971. Evaluation of the worth of additional data. Water Resources Bulletin, Vol. 7, No. 4, pp. 700-707.

Dawdy, D. R., 1979. The worth of hydrologic data. Water Resources Research, Vol. 15, No. 6, pp. 1726-1732.

Dawdy, D. R., Kubik, H. E. and Close, E. R., 1970. Value of streamflow data for project design - A plot study. Water Resources Research, Vol. 6, No. 4, pp. 1045-1050.

Duckstein, L., Davis, D. R. and Boganti, I., 1974. Applications of decision theory to hydraulic engineering. paper presented at ASCE Hydraulic Division Specialty Conference, American Ass. of Civil Engineers, Stanford, California.

Eagleson, P., 1967. Optimum design and rainfall networks. Water Resources Research, Vol. 3, No. 5, pp. 1021-1033.

Fiering, M. B., 1965. An optimization scheme for gaging. Water Resources Research, No. 1, Vol. 4, pp. 463-470.

Gradshtein, I. S. and Ryzhik, I. M., 1980. Table of Integrals, Series and Products. Academic Press, Inc., New York.

Harmancioglu, N., 1980. Measuring the information content of hydrological processes by the entropy concept (in Turkish). Ege University, Faculty of Civil Engineering, Ph.D. Thesis on Hydrology and Hydraulic Works, Izmir, No. 4, 164 p.

Jaynes, E. T., 1978. Where do we stand on maximum entropy? in E. T. Jaynes: Papers on probability, statistics and statistical physics, edited by Rosenkratz, D. Reidel Publishing Co., 1983.

Jaynes, E. T., 1982. On the old rationale of maximum entropy methods. Proceedings of the IEEE, Vol. 70, No. 9, pp. 939-952.

Klemes, V., 1977. Value of information in reservoir optimization. Water Resources Research, Vol. 13, No. 5, pp. 837-850.

Langbein, W. B., 1954. Stream gaging networks. International Assoc. Scientific Hydrology Publication 38, pp. 293-303.

Lenton, R. L. and Rodriguez-Iturbe, 1977. Rainfall network system analysis: the optimal estimation of total areal storm depth. Water Resources Research, Vol. 13, No. 5, pp. 825-836.

Matalas, N. C. and Gilroy, E. J., 1968. Some comments on regionalization in hydrologic studies. Water Resources Research, Vol. 4, No. 6, pp. 1361-1369.

Moss, M. E., 1970. Optimum operating procedure for a river gaging station established to provide data for design of a water supply project. Water Resources Research, Vol. 6, No. 4, pp. 1051-1061.

Moss, M. E. and Dawdy, D. R., 1971. An optimum path to reservoir design based on a worth of data. in Proceedings of the Water Resources Symposium, pp. DG-1 - DG-6, Indian Institute of Science, Bangalore, India.

Moss, M. E. and Karlinger, M. R., 1974. Surface water network design by regression analysis simulations. Water Resources Research, Vol. 10, No. 3, pp. 427-433.

Rodriguez-Iturbe, I. and Mejia, J. M., 1974. The design of rainfall networks in time and space. Water Resources Research, Vol. 10, No. 4, pp. 713-728.

Shannon, C. E., 1948. A mathematical theory of communication. Bell System Technical Journal, Vol. 27, pp. 379-423.

Singh, V. P. and Krstanovic, P. F., 1985. A stochastic model for sediment yield. submitted to Water Resources Research.

Singh, V. P., Singh K., and Rajagopal, A. K., 1985. Application of the principle of maximum entropy (POME) to hydrologic frequency analysis. Louisiana Water Resources Research Institute, Louisiana State University, Baton Rouge, LA.

Slack, J. R., Wallis, J. R. and Matalas, N. C., 1975. On the value of information to flood frequency analysis. Water Resources Research, Vol. 11, No. 5, pp. 629-647.

Sonuga, J. O., 1976. Entropy principle applied to rainfall-runoff process. Journal of Hydrology, Vol. 30, pp. 81-94.

Thomas, D. M. and Benson, M. A., 1970. Generalization of streamflow characteristics from drainage-basin characteristic. U.S. Geological Survey Water Supply Paper, 1975, 55 pp.

APPENDIX A

The pdf for the random sampling of distances d can be derived as follows. Given the constraints:

$$\int_{-\infty}^{+\infty} p(d)\ d(d) = 1 \tag{A-1}$$

$$\int_{-\infty}^{+\infty} d\ p(d)\ d(d) = \bar{d} \tag{A-2}$$

$$\int_{-\infty}^{+\infty} d^2\ p(d)\ dd = E(d^2) = \bar{d}^2 + S_d^2 \tag{A-3}$$

The pdf by the maximum entropy method is given as:

$$p(d) = \exp(-\lambda_0 - \lambda_1 d - \lambda_2 d^2) \tag{A-4}$$

From (A-1),

$$\int_{-\infty}^{+\infty} \exp(-\lambda_0 - \lambda_1 d - \lambda_2 d^2)\ dd = 1$$

$$\exp(\lambda_0) = \int_{-\infty}^{+\infty} \exp(-\lambda_1 d - \lambda_2 d^2)\ dd \tag{A-5}$$

Simplifying,

$$\exp(\lambda_0) = \exp(\frac{\lambda_1^2}{4\lambda_2}) \int_{-\infty}^{+\infty} \exp(-(\lambda_2 d)^{0.5} + \frac{\lambda_1}{2(\lambda_2)^{0.5}})^2\ dd$$

and integrating,

$$\exp(\lambda_0) = (\frac{\pi}{\lambda_2})^{0.5} \exp(\frac{\lambda_1^2}{4\lambda_2})$$

The zeroth Lagrange multiplier is:

$$\lambda_0 = \frac{1}{2} \ln \pi - \frac{1}{2} \ln \lambda_2 + \frac{\lambda_1^2}{4\lambda_2} \tag{A-6}$$

From (A-5),

$$\lambda_0 = \ln[\int_{-\infty}^{+\infty} \exp(-\lambda_1 d - \lambda_2 d^2)\ dd] \tag{A-7}$$

Differentiating (A-5) with respect to λ_1 and λ_2,

$$\frac{\partial\lambda_0}{\partial\lambda_1} = - \frac{\int_{-\infty}^{+\infty} d \exp(- \lambda_1 d - \lambda_2 d^2)\, dd}{\int_{-\infty}^{+\infty} \exp(- \lambda_1 d - \lambda_2 d^2)\, dd} = - \int_{-\infty}^{+\infty} d\, f(d)\, dd = - \bar{d} \qquad \text{(A-8)}$$

$$\frac{\partial\lambda_0}{\partial\lambda_2} = - \frac{\int_{-\infty}^{+\infty} d^2 \exp(- \lambda_1 d - \lambda_2 d^2)\, dd}{\int_{-\infty}^{+\infty} \exp(- \lambda_1 d - \lambda_2 d^2)\, dd} = - \int_{-\infty}^{+\infty} d^2 f(d)\, dd$$

$$= - (S_d^2 + \bar{d}_d^2) \qquad \text{(A-9)}$$

Differentiating (A-5),

$$\frac{\partial\lambda_0}{\partial\lambda_1} = \frac{\lambda_1}{2\lambda_2} \qquad \text{(A-10)}$$

$$\frac{\partial\lambda_0}{\partial\lambda_2} = - \frac{1}{2\lambda_2} - \frac{\lambda_1}{4\lambda_2^2} \qquad \text{(A-11)}$$

Equating (A-8) to (A-10) and (A-9) to (A-11) and solving for λ_1 and λ_2,

$$\lambda_1 = - \frac{\bar{d}}{S_d^2} \qquad \text{(A-12)}$$

$$\lambda_2 = \frac{1}{2S_d^2} \qquad \text{(A-13)}$$

The pdf by the maximum entropy method is,

$$p(d) = \frac{1}{(2\pi S_d^2)^{0.5}} \exp[- \frac{(d - \bar{d})^2}{2S_d^2}] \qquad \text{(A-14)}$$

APPENDIX B

Derivation of r(d) is performed as follows. Specifying constraints for exponential distribution,

$$\int_{0}^{+\infty} f(x)\ dx = 1 \qquad \text{(B-1)}$$

$$\int_{0}^{\infty} x\ f(x)\ dx = \bar{x} \qquad \text{(B-2)}$$

The pdf of maximum entropy takes the form:

$$f(x) = \exp(-\ \lambda_0 - \lambda_1 x) \qquad \text{(B-3)}$$

From (B-1),

$$\int_{0}^{\infty} \exp(-\ \lambda_0 - \lambda_1 x)\ dx = 1$$

$$\exp(\lambda_0) = \int_{0}^{\infty} \exp(-\ \lambda_1 x)\ dx \qquad \text{(B-4)}$$

and

$$\lambda_0 = -\ \ln\ \lambda_1 \qquad \text{(B-5)}$$

From (B-5),

$$\frac{\partial \lambda_0}{\partial \lambda_1} = -\ \frac{1}{\lambda_1} \qquad \text{(B-6)}$$

From (B-4),

$$\frac{\partial \lambda_0}{\lambda \partial_1} = -\ \frac{\int_{0}^{\infty} \exp(-\ \lambda_1 x)\ dx}{\int_{0}^{\infty} x\ f(x)\ dx} = -\ \int^{\infty} x\ f(x)\ dx = -\ \bar{x} \qquad \text{(B-7)}$$

By equating (B-6) and (B-7), λ_1 is obtained:

$$\lambda_1 = \frac{1}{\bar{x}} \qquad \text{(B-8)}$$

Substituting (B-8) and (B-5) into (B-3),

$$f(x) = \frac{1}{\bar{x}}\ \exp(-\ \frac{x}{\bar{x}})$$

or

$$r(d) = \frac{1}{\bar{d}}\ \exp(-\ \frac{d}{\bar{d}}) \qquad \text{(B-9)}$$

APPENDIX C

The entropy method for a time series where information is given in terms of autocovariances can be derived as follows. Given:

$$c_0(x_0, \ldots, x_T) = \frac{1}{T+1} \sum_{j=0}^{T} x_j^2 \qquad \text{(variance)}$$

$$c_k(x_0, \ldots, x_T) = \frac{1}{T+1} \sum_{j=0}^{T} x_j \, x_{j-k} \qquad \text{(autocovariance)} \tag{C-1}$$

If no other information is available but (C-1), the probability density function that has maximum entropy while agreeing with the data will yield the set of $\{\lambda_0, \ldots, \lambda_m\}$ where $0 \le k \le m$ represents the number of available constraints. The maximum entropy distribution for this case is

$$p(x_0, \ldots, x_T) = \frac{1}{Z} \exp[- \sum_{k=0}^{m} \lambda_k \, 0.5 \, r_k] \tag{C-2}$$

where Z is the partition function from general POME procedure and is determined from:

$$\int_{-\infty}^{+\infty} p(x_0, \ldots, x_T) \, dx_0 \, dx_T = 1 \tag{C-3}$$

Note that (C-2) can be expressed as:

$$p(x_0, \ldots, x_T) = \frac{1}{Z} \exp[- 0.5(X^T \cdot \Lambda \cdot X)] \tag{C-4}$$

where Λ is a matrix

$$\Lambda_{ij} = \begin{cases} \lambda_{j-i} \, , & |j-i| \le m \\ 0 \, , & \text{otherwise} \end{cases} \tag{C-5}$$

and X^T,X are vector-transpose and vector of rainfall depths. The extension of this solution (Jaynes, 1982) considers the case when the given time base T is much greater than the number of constraints m. From the Toeplitz theory for T >> m, the eigenvalues of Λ will become:

$$g_j = g(z_j) \qquad \text{(C-6)}$$

where z_j are the roots of $z^{T+1} = 1$ on the unit circle or:

$$z_j = \exp[2\pi ij/(T+1)] \qquad \text{(C-7)}$$

where $0 \leq j \leq T$. Since the partition function is given as:

$$\ln Z = -0.50 \sum_{j=0}^{T} \ln g_j + \text{constant} \qquad \text{(C-8)}$$

Then

$$\ln Z \to -0.50 \sum_{j=0}^{T} \ln[g(z_j)]$$

where $g(z_j)$ is defined as:

$$g(z_j) = \sum_{k=0}^{m} \lambda_k z^k \qquad \text{(C-9)}$$

As $T \to \infty$, $\ln Z \to -0.50 \sum_{j=0}^{\infty} \ln[g_j(\exp(2\pi ij/(N+1)))]$ (C-10)

This summation for longer time interval becomes integration on a unit circle and z_j will be close to one another, then

$$\frac{2}{T+1} \ln Z(\lambda_k) = -\frac{1}{2\pi} \int_0^{2\pi} \ln[g(\exp(i\theta)] \, d\theta$$

or

$$\ln Z(\lambda_k) = -\frac{T+1}{4\pi} \int_0^{2\pi} \ln[\sum_{k=0}^{m} \lambda_k \exp(ik\theta)] \, d\theta \qquad \text{(C-11)}$$

Then the Lagrangian multipliers are determined from

$$0.5(T+1)\, c_k = -(\partial/\partial\lambda_k)\,[\ln Z(\lambda_k)] \qquad \text{(C-12)}$$

or

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp(ik\theta)\, d\theta}{\sum_{k=0}^{m} \lambda_k \exp(ik\theta)} \qquad \text{(C-13)}$$

To illustrate, let us work out a couple of simple examples for various values of k.

(a) For k = 0:

$$c_0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\lambda_0} = \frac{1}{2\pi} 2\pi \frac{1}{\lambda_0}$$

which yields

$$\lambda_0 = 1/c_0 \tag{C-14}$$

This term can also be obtained from

$$c_k = 1/(T+1)\ \partial/\partial\lambda_k \ln(\det \Lambda) \tag{C-15}$$

where $\det \Lambda = \lambda_0^{T+1}$, so

$$\frac{\partial}{\partial\lambda_0} [\ln(\lambda_0^{T+1})] = \frac{(T+1)\lambda_0^T}{\lambda_0^{T+1}} = \frac{T+1}{\lambda_0}$$

Thus,

$$c_0 = \frac{1}{T+1} \frac{T+1}{\lambda_0} = \frac{1}{\lambda_0}$$

which is the same as (C-14).

(b) For k = 1

$$c_0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\lambda_0 + \lambda_1 \exp(i\theta)}$$

$$c_1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp(i\theta)d\theta}{\lambda_0 + \lambda_1 \exp(i\theta)} \tag{C-16}$$

Using the substitutions: $z = \exp(i\theta)$ and $dx = d\theta i \exp(i\theta)$,

$$c_0 = \frac{1}{2\pi\lambda_1} \int_{|z|=1} \frac{dz}{z[z - (-\lambda_0/\lambda_1)]} = \frac{1}{2\pi i\lambda_1} \int_{|z|=1} \frac{dz}{z(z - z_0)}$$

$$c_1 = \frac{1}{2\pi\lambda_1} \int_{|z|=1} \frac{dz}{z - z_0}, \text{ where } z_0 = -\lambda_0/\lambda_1$$

The first equation can be solved by residue theorem, while the second equation by Cauchy integral formula, yielding:

$$c_1 = (1/2\pi i\lambda_1) \cdot 2\pi, \text{ if } (z = z_0), \text{ where } f(z) = 1$$

$$c_1 = (1/\lambda_1) \cdot 1 = 1/\lambda_1$$

or $\lambda_1 = 1/c_1$ (C-17)

which is the new Lagrangian multiplier introduced in the model after considering both c_0 and c_1.

(c) For k = 2, autocovariances of rainfall depth are given as:

$$c_0 = \frac{1}{T+1} \sum_{j=0}^{T} x_j^2$$

$$c_1 = \frac{1}{T+1} \sum_{j=0}^{T-1} x_j \; x_{j+1} \qquad \text{(C-18)}$$

$$c_2 = \frac{1}{T+1} \sum_{j=0}^{T-2} x_j \; x_{j+2}$$

The Lagrangian multipliers are then determined from:

$$c_0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\lambda_0 + \lambda_1 \exp(i\theta) + \lambda_2 \exp(2i\theta)}$$

$$= \frac{1}{2\pi i \lambda_2} \int_{|z|=1} \frac{dz}{z(z - z_1)(z - z_2)}$$

$$c_1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp(i\theta) d\theta}{\lambda_0 + \lambda_1 \exp(i\theta) + \lambda_2 \exp(2i\theta)}$$

$$= \frac{1}{2\pi i \lambda_2} \int_{|z|=1} \frac{dz}{(z - z_1)(z - z_2)}$$

$$c_2 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp(2i\theta) d\theta}{\lambda_0 + \lambda_1 \exp(i\theta) + \lambda_2 \exp(2i\theta)}$$

$$= \frac{1}{2\pi i \lambda_2} \int_{|z|=1} \frac{z dz}{(z - z_1)(z - z_2)}$$

Solving the last equation:

$$c_2 = (1/\lambda_2) \cdot [\text{Res}(z = z_1) + \text{Res}(z = z_2)]$$

$$\text{Res}(z = z_1) = \lim_{z \to z1} \left[(z - z_1) \cdot \frac{z}{(z - z_1)(z - z_2)}\right] = \frac{z_1}{z_1 - z_2}$$

$$\text{Res}(z = z_2) = \lim_{z \to z2} \left[(z - z_2) \cdot \frac{z}{(z - z_1)(z - z_2)}\right] = \frac{z_2}{z_1 - z_2}$$

Thus:

$$c_2 \;] \; (1/\lambda_2) \; [z_1/(z_1 - z_2) + z_2/(z_2 - z_1)] = 1/\lambda_2$$

$$\lambda_2 = 1/c_2 \qquad \text{(C-19)}$$

A similar derivation for c_0, c_1, c_2, c_3 yields

$$\lambda_3 = 1/c_3$$

or in general, the last Lagrangian multiplier introduced to the model will always be as:

$$\lambda_k = 1/c_k \qquad \text{(C-20)}$$

Exact theoretical proof can be done simply by mathematical induction.

APPENDIX D

Levinson-Burg algorithm for solving the Toeplitz matrix

Given the matrix equation of either autocovariance c_k or correlation coefficients r_k:

$$\begin{bmatrix} r_0 & r_{-1} & r_{-2} & \cdots & r_{1-m} \\ r_1 & r_0 & \cdots & \cdots & \cdot \\ \vdots & \vdots & & & \vdots \\ r_{m-1} & \cdots & \cdots & r_1 & r_0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ b_1 \\ \vdots \\ b_{m-1} \end{bmatrix} = \begin{bmatrix} P_{m-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{(D-1)}$$

For known values of r_k, P_{m-1} and b_m are solved as a system of m equations with m unknowns. To initiate the algorithm one has to start with m = 1. Then $r_0 = P_0$ and $b_0 = 1$. Every higher m (≥ 2) system is solved by using the solution of previous one with lower m. In general, the system (m+1)x(m+1) will be solved from (D-1) as:

$$\begin{bmatrix} r_0 & r_{-1} & \cdots & r_{-m} \\ r_1 & r_0 & \cdots & r_{1-m} \\ \vdots & & & \vdots \\ r_{m-1} & & & r_{-1} \\ r_m & \cdots & \cdots & r_0 \end{bmatrix} = \begin{bmatrix} 1 \\ b_1 \\ \vdots \\ b_{m-1} \\ 0 \end{bmatrix} + d_m \begin{bmatrix} 0 \\ b^*_{m-1} \\ \vdots \\ b^*_1 \\ 1 \end{bmatrix} = \begin{bmatrix} P_{m-1} \\ 0 \\ \vdots \\ 0 \\ \Delta_m \end{bmatrix} + d_m \begin{bmatrix} \Delta^*_m \\ 0 \\ \vdots \\ 0 \\ P_{m-1} \end{bmatrix}$$

From this equation:

$$\left. \begin{aligned} \Delta_m &= \sum_{n=0}^{m-1} r_{m-n}\, b_m \\ \text{and} \quad d_m &= -\frac{\Delta_m}{P_{m-1}} \end{aligned} \right\} \qquad \text{(D-2)}$$

The matrix of (m+1)x(m+1) system will have the following coefficients:

$$\begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_{m-1} \\ a_m \end{bmatrix} = \begin{bmatrix} 1 \\ b_1 \\ \vdots \\ b_{m-1} \\ 0 \end{bmatrix} + d_m \begin{bmatrix} 0 \\ b^*_{m-1} \\ \vdots \\ b^*_1 \\ 1 \end{bmatrix} \qquad \text{(D-3)}$$

Note that for the real values of precipitation depths, $b_m = b^*_m$ which simplifies calculation in (D-2) and (D-3). From (D-3)

$$a_s = b_s + d_{m-s} \tag{D-4}$$

and new constant P_m is solved by using P_{m-1} and d_m:

$$P_m = P_{m-1} (1 - |d_m|^2) \tag{D-5}$$

Thus, the new system of (m+1)x(m+1) is defined as:

$$\begin{bmatrix} r_0 & \cdots & r_{-m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ r_m & \cdots & r_0 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \cdot \\ \cdot \\ r_0 \end{bmatrix} = \begin{bmatrix} P_m \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{D-6}$$

Equation (D-6) is the starting point in solving (m+2)x(m+2) system of equations, after substitution $b_k = a_k$ and $P_{m-1} = P_m$.