Fundamentals of Engineering Review for Dynamics

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Review Notes available in PDF format @ http://www.lsu.edu/eng/docs/FE-Exam-Review/Dynamics.pdf


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Dynamics Problem Decomposition

Dynamics

Kinematics

Geometric descriptions of motion & constraints

Kinetics

Loading relationships which dictate CHANGES in motion
Dynamic Studies

Dynamics:
(Kinematics & Kinetics)

Particles

Rigid Bodies

Plane motion:
DOF (Degrees of Freedom)?

- m – mass (inertia)
- \( \textbf{P} \) - position \{\((x, y), (r, \theta)\)\} (2DOF)
- \( \textbf{V} \) – velocity \{\((v_x, v_y), (v_r, v_\theta), (\theta, v)\)\}
- \( \textbf{A} \) – acceleration
  \{\((a_x, a_y), (a_r, a_\theta), (a_n, a_t)\)\}
- \( m \) & \( I \) – add in rotational inertia \((I)\)
- \((\textbf{P}, \theta)\) - position & orientation (3DOF)
- \( \textbf{V} \) – velocity \{\(\textbf{V}_A, \omega\)\}
- \( \textbf{A} \) – acceleration \{\(\textbf{A}_A, \alpha\)\}
Getting Started => Particle Kinematics

- Rectilinear Motion
  - Movement along a straight line in 1-2 or 3D
    - 1 Degree of Freedom (DOF)* - $s(t)$

- Curvilinear Motion
  - Movement of particle along an arbitrary path through space
Rectilinear Motion Overview (Calculus/Physics Review!):

- **Position** - \( s(t) \)
- **Speed** - \( v(t) \)

(1) \[ v = \frac{ds}{dt} = s \]

- **Acceleration** - \( a(t) \)

(2) \[ a = \frac{dv}{dt} = v = \frac{d^2s}{dt^2} = s \]

- **Typical Functions ??**
  - Polynomial, Trigonometric, Logarithmic, Exponential
Rectilinear Motion Summary:

- **Position** - \( s(t) \)
- **Speed** - \( v(t) \)
  
  \[
  (1) \quad v = \frac{ds}{dt} = \dot{s}
  \]

- **Acceleration** - \( a(t) \)
  
  \[
  (2) \quad a = \frac{dv}{dt} = \ddot{v} = \frac{d^2s}{dt^2} = \ddot{s}
  \]

- **Alternate form?**
  
  \[
  (2^*) \quad v \ dv = a \ ds
  \]

\( a(t) \) \( \Rightarrow \) Solid Rocket Propulsion  \( a(v) \) \( \Rightarrow \) aerodynamic drag

\( a(s), \ v(s) \) \( \Rightarrow \) Gravitational fields, springs, conservative forces etc.

\( (s, v, a \ & \ t) \) \( \Rightarrow \) \( t \) independent parameter
Given:  \( s = 2t^2 - 8t + 3 \)

Find:  Displacement from \( t = 1 \) to \( t = 3 \)
Distance traveled from \( t = 1 \) to \( t = 3 \)

\[
\begin{align*}
  s &= 2t^2 - 8t + 3 \\
  v &= \dot{s} = 4t - 8 \\
  a &= \ddot{s} = 4
\end{align*}
\]

\[
\begin{align*}
  s(1) &= -3 & s(3) &= -3 & s(2) &= -5 \\
  v(2) &= 0 \\
  a(2) &= 4
\end{align*}
\]

Since, \( s(3) - s(1) = 0 \)  \( \Rightarrow \)  displacement = 0

Reversal @ \( v\big|_{t=2} = 0 \)  distance traveled = \(| s(2) - s(1) | + |s(3) - s(2)| \)

\[
\begin{align*}
  &= | -5 - (-3) | + | (-3) - (-5) | \\
  &= 2 + 2 = 4
\end{align*}
\]
Rectilinear Kinematics: Accel. a function of velocity – $a(v)$

Given:
- A freighter moving at 8 knots when engines are stopped
- Deceleration $a = -kv^2$
- Speed reduces to 4 knots after ten minutes

Find:
(A) Speed of the ship as a function of time $v(t)$
(B) How far does the ship travel in the 10 minutes it takes to reduce the speed by 1/2?

Solution:
(A) With $a$, $v$ & $t$ parameters given/requested, use $a=dv/dt$ form

$$ a(v) = \frac{dv}{dt} \Rightarrow dt = \frac{dv}{a(v)} \Rightarrow \int_{t_i}^{t_f} dt = \int_{v_i}^{v_f} \frac{dv}{-kv^2} $$

$$ t_f = t(v) = \int_{v_i}^{v_f} \frac{dv}{a(v)} - t_i = \int_{8}^{v_f} \frac{dv}{-kv^2} + 0 $$

$$ \Rightarrow t_f = \left. \frac{1}{kv} \right|_{8}^{v_f} = \frac{1}{k} \left( \frac{1}{v_f} - \frac{1}{8} \right) \Rightarrow v_f = v(t_f) = \frac{8}{8kt_f + 1} \text{ (knots)} $$
Rectilinear Kinematics: Accel. a function of velocity – \(a(v)\)

- Substituting BC’s helps resolve the unknown constant \(k\)

\[
t = \frac{10 \text{ (min)}}{60 \text{ (min/hr)}} = \frac{1}{6} \text{ hr} , \quad v = 4 \text{ knots}
\]

\[
\Rightarrow v(1/6) = \frac{8}{8k(1/6) + 1} = 4 \text{ (knots)} \Rightarrow k = \frac{3}{4} \left( \frac{1}{\text{nm}} \right)
\]

and the resulting expression for speed of the ship as a function of time \(v(t)\) is as follows

\[
v_f = v(t) = \frac{8}{6t + 1} \text{ (knots)}
\]

- From here, there are two alternatives for resolving the second question
Rectilinear Kinematics: Accel. a function of velocity – $a(v)$

(B) METHOD 1: Now, knowing the velocity as a function of time

$$v(t) = \frac{8}{6t + 1} = \frac{ds}{dt}$$

the boat’s position can be found by integration

$$\int_0^{s_f} ds = \int_0^{t_f} \frac{8}{6t + 1} dt$$

$$s_f - 0 = \frac{4}{3} \ln(6t + 1) \bigg|_0^{t_f} = \frac{4}{3} \left( \ln(6t + 1) - \ln(1) \right)$$

and the resulting expression for position of the ship as a function of time $s(t)$

$$s_f = s(t) = \frac{4}{3} \ln(6t + 1)$$

can now be used to find the particular displacement/distance at $t=1/6$ hr!

$$s(1/6) = \frac{4}{3} \ln(6(1/6) + 1) = \frac{4}{3} \ln(2) \text{ (nautical miles)}$$
Rectilinear Kinematics: Accel. a function of velocity – a(v)

(B) METHOD 2: With a, v & s parameters given/requested, use $ads=vdv$ form

$$s_f = s(v) = \int_{v_i}^{v_f} \frac{vdv}{a(v)} + s_i \Rightarrow s(4) = \int_{8}^{4} \frac{vdv}{-3/4v^2} + 0$$

and the boat’s displacement (position?) can again be found by integration

$$s(4) = \frac{-4}{3} \int_{8}^{4} \frac{dv}{v} = \frac{-4}{3} \ln v\bigg|_{8}^{4} = \frac{-4}{3} (\ln 4 - \ln 8) = \frac{4}{3} \ln \frac{8}{4}$$

and as was seen before

$$s(t = 1/6) \Rightarrow s(v = 4) = \frac{4}{3} \ln(2) \quad \text{(nautical miles)}$$

Q.E.D.
2D Curvilinear Kinematics Summary:

- **Position**
  \[ r(t) = x(t)i + y(t)j \]
  \[ = r(t)e_r \]
  \[ ? \text{path?} \]

- **Velocity**
  \[ v(t) = \dot{r}(t) = x \dot{i} + y \dot{j} \]
  \[ = ve_t = s e_t \]
  \[ = re_r + r \theta e_\theta \]

- **Acceleration**
  \[ a(t) = \ddot{v}(t) = \ddot{r}(t) = x \dot{i} + y \dot{j} \]
  \[ = se_t + \rho \theta^2 e_n = ve_t + \frac{v^2}{\rho} e_n \]
  \[ = \left( r - r \theta^2 \right) e_r + \left( r \theta + 2 r \theta \right) e_\theta \]
2D Curvilinear Motion: Coordinates & Conversions

- Cartesian <-> Polar <-> Path

\[ \mathbf{r}(t) = x \mathbf{i} + y \mathbf{j} = r \mathbf{e}_r \]
\[ \mathbf{e}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} = \frac{x}{r} \mathbf{i} + \frac{y}{r} \mathbf{j} \]
\[ \mathbf{e}_\theta = \mathbf{k} \times \mathbf{e}_r = \cos \theta \mathbf{j} - \sin \theta \mathbf{i} \]
\[ \mathbf{i} = \mathbf{cos} \, \mathbf{e}_r - \mathbf{sin} \, \mathbf{e}_\theta \]
\[ \mathbf{j} = \mathbf{k} \times \mathbf{i} = \mathbf{cos} \, \mathbf{e}_\theta + \mathbf{sin} \, \mathbf{e}_r \]

\[ \mathbf{v}(t) = \dot{\mathbf{r}}(t) = x \dot{\mathbf{i}} + y \dot{\mathbf{j}} = v \mathbf{e}_t = s \mathbf{e}_t \]
\[ \mathbf{e}_t = \frac{\mathbf{v}}{\lVert \mathbf{v} \rVert} = \frac{x}{v} \mathbf{i} + \frac{y}{v} \mathbf{j} \]
\[ \mathbf{e}_n = \mathbf{k} \times \mathbf{e}_t = \frac{x}{v} \mathbf{j} - \frac{y}{v} \mathbf{i} \]

\[ \mathbf{i} = \cos \Psi \mathbf{e}_t - \sin \Psi \mathbf{e}_n \]
\[ \mathbf{j} = \mathbf{k} \times \mathbf{i} = \cos \Psi \mathbf{e}_n + \sin \Psi \mathbf{e}_t \]
Curvilinear Motion: Cartesian Coordinates

- Projectile Motion
  - Scale w.r.t. earth such that gravity $g$ is \textit{constant}
    - $|g| = 32.2 \text{ ft/s}^2 = 9.81 \text{ m/s}^2$
  - Neglect any air resistance
  - Motion is PARABOLIC thus PLANAR!
  - Typically align
    - $y$-axis along gravity vector
    - $x$-axis horizontal in direction of motion
      - $a(t) = 0\mathbf{i} - g\mathbf{j} = [0, -g]$
        - $z$ component drops out!
  - Integrate rectilinear relations
    - Two (2) scalar relations
    - One VECTOR relationship
Curvilinear Motion: Projectile Motion

- Typical P.M. queries
  - Max Height
  - Max Range
  - Time @ some place along trajectory
  - Later w/ Path & Polar Coord
    - Velocity (speed, direction/tangent)
    - Curvature, rate of speed change ....

\[
a(t) = 0\mathbf{i} - g\mathbf{j} = [0, -g]
\]

\[
\Rightarrow \mathbf{v}_f = a(t_f - t_i) + \mathbf{v}_i
\]

\[
x_f = v_{x_i}(t_f - t_i) + x_i
\]

\[
y_f = \frac{-g}{2}(t_f - t_i)^2 + v_{y_i}(t_f - t_i) + y_i
\]

- Reconsider problems w/ different axes placement/orientation
Given: launch at 3600 m altitude \( v_o = 180 \text{ m/s} \) angle 30°

\[
\begin{align*}
\ddot{x} &= 0 \\
\dot{x} &= 180 (\cos 30) = 156 \\
x &= 156 t \\
\ddot{y} &= -9.81 \\
\dot{y} &= 180 (\sin 30) - 9.81 t = 90 - 9.81 t \\
y &= 90 t - 4.905 t^2 + 3600 \\
\text{for } h \text{ set } \dot{y} &= 0 \quad \Rightarrow \quad t = 9.17 \quad \Rightarrow \quad h = y = 4013 \text{ m} \\
\text{for } t_T \text{ set } y &= 0 \quad \Rightarrow \quad t = 37.8
\end{align*}
\]
Curvilinear Kinematics: Projectile Motion example

Given:
- Figure shown w/ ground \( y = -kx^2 \)
- \( t_0=0, (x_0,y_0)=0, v_0 = v_0 @ \theta \) above horizon

Find:  In terms of \( v_0, \theta \) & \( k \)
- (A) The location at impact \((x_i,y_i)\)
- (B) Velocity & Speed @ impact, \( v_f, v_i \)
- (C) Elapsed time @ impact, \( t_i \)

Solution:
- 2D projectile motion
- Get expressions for \( v_x(t), v_y(t) \) then \( x(t), y(t) \)

\[
\Rightarrow \frac{v_f(t)}{r_f} = a_c (t_f - t_i) + v_i = \frac{a_c}{2} (t_f - t_i)^2 + v_i (t_f - t_i) + r_i
\]

- Substitute into ground constraint expression
  - Solve for time of impact \((t_i)\)
  - With \( t_i \) known, substitute & solve for \((x_i,y_i)\)
Curvilinear Kinematics: Projectile Motion

- **IC’s** => \( t_0 = 0, (x_0, y_0) = 0, v_0 = v_0 \ @ \ \theta \)

\[
\mathbf{a}(t) = 0\mathbf{i} - g\mathbf{j} = [0, -g]
\]

\[
\Rightarrow (B) \quad \mathbf{v}_I = \mathbf{a}(t_I - 0) + \mathbf{v}_0 = [v_{x_I}, v_{y_I}]
\]

\[
v_{x_I} = v_0 \cos \theta
\]

\[
v_{y_I} = -gt_I + v_0 \sin \theta
\]

- **Speed**

\[
s = v = \sqrt{v_{x_I}^2 + v_{y_I}^2} = \sqrt{(v_0 \cos \theta)^2 + (-gt_I + v_0 \sin \theta)^2}
\]

\[
= \sqrt{v_0^2 - 2gv_0 \sin \theta t_I + (gt_I)^2}
\]
Curvilinear Kinematics: Projectile Motion

\[ \Rightarrow (B) \quad \mathbf{v}_I = \mathbf{a}(t_I - 0) + \mathbf{v}_0 = \left[ v_{x_I}, v_{y_I} \right] \]

\[ \Rightarrow (A) \quad \mathbf{r}_I = \frac{\mathbf{a}}{2} (t_I - 0)^2 + \mathbf{v}_0 (t_I - 0) + \mathbf{0} \]

\[
\begin{align*}
x_I &= v_0 \cos \theta \cdot t_I \\
y_I &= \frac{-g}{2} t_I^2 + v_0 \sin \theta \cdot t_I \\
y &= -kx^2
\end{align*}
\]

\[ \frac{-g}{2} t_I^2 + v_0 \sin \theta \cdot t_I = -k \left( v_0 \cos \theta \cdot t_I \right)^2 \]

\[ \Rightarrow (C) \quad t_I = \frac{2v_0 \sin \theta}{g - 2k(v_0 \cos \theta)^2}, \quad t_I = 0 \]

- Substitute value for \( t_I \) into position, velocity & speed relations for solution
Path Coord. Example ref: Meriam&Kraige 2-8

Given:
- A rocket at high altitude with
- \( \mathbf{a}_0 = 6\mathbf{i} - 9\mathbf{j} \) (m/s\(^2\))
- \( \mathbf{v}_0 = 20 \) (km/hr) @ 15 below horizontal

Find: At instant given
(A) The normal & tangential accelerations
(B) Rate at which speed is increasing
(C) Radius of curvature of the path
(D) Angular rotation rate of the radial from CG to center of curvature

Solution:
- "High altitude" means negligible air resistance
- Interested only at this instant (NO Integration required)
- Cartesian specified, asking for Path coord parameters
- \( \mathbf{V} \) given is TANGENT TO THE PATH
  - Use this to relate path to cartesian coordinates
Solution (cont'd):

\[ e_t = \frac{v}{|v|} = \frac{v}{v} = \cos 15^\circ \mathbf{i} - \sin 15^\circ \mathbf{j} \]
\[ e_n = \pm (\mathbf{k} \times e_t) \quad (2D \text{ shortcut!}) \]
\[ = -\cos 15^\circ \mathbf{j} - \sin 15^\circ \mathbf{i} \]

(A) \( a_n \) & \( a_t \) =?

\[ |a_t| = a \cdot e_t = (6\mathbf{i} - 9\mathbf{j}) \cdot \left( \cos 15^\circ \mathbf{i} - \sin 15^\circ \mathbf{j} \right) = 8.12 \text{ (m/s}^2\text{)} = a_t \]
\[ |a_n| = a \cdot e_n = (6\mathbf{i} - 9\mathbf{j}) \cdot \left( -\cos 15^\circ \mathbf{j} - \sin 15^\circ \mathbf{i} \right) = 7.14 \text{ (m/s}^2\text{)} = a_n \]

(B) \( \dot{v} = ?? \quad \dot{v} = |a_t| = 8.12 \text{ (m/s}^2\text{)} \)

(C) \( \rho =? \quad a_n = \frac{v^2}{\rho} \Rightarrow \rho = \frac{v^2}{a_n} = \frac{(20 \text{ km/hr})^2}{7.14(\text{m/s}^2)} \left( \frac{1 \text{ hr}}{3600 \text{s}} \ast \frac{10^3 \text{ m}}{\text{km}} \right)^2 = 4.32 \text{ (10}^6 \text{) m} \)
Solution (cont’d):

(D) \( \dot{\theta} = \) ??

- Look either at \( a_n \) or velocity

\[ a_n = \rho \dot{\theta} \]

\[ \Rightarrow \dot{\theta} = \sqrt{\frac{a_n}{\rho}} = \sqrt{\frac{7.14 \text{ (m/s}^2)\, }{4.32 \times 10^6 \text{(m)}}} = 12.9 \times 10^{-4} \frac{1}{s} \]

\( v = \rho \dot{\theta} \)

\[ \Rightarrow \dot{\theta} = \frac{v}{\rho} = \frac{20 \text{ km/hr}}{4.32 \times 10^6 \text{(m)}} \left( \frac{1 \text{ hr}}{3600 \text{s}} \times \frac{10^3 \text{ m}}{\text{km}} \right) = 12.9 \times 10^{-4} \text{s}^{-1} \]
**RELATIVE MOTION**

\[ \mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \]
\[ \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \]
\[ \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \]

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**Special Case: Rigid Bodies**

When A & B are two points on the same rigid body:
- the relative motion is circular
- \( \mathbf{v}_{B/A} \) is perpendicular (\( \perp \)) to \( \mathbf{r}_{B/A} \)
  & \( | \mathbf{v}_{B/A} | = | \mathbf{\omega}_{AB} \ AB | \)

\[ \mathbf{v}_B = \mathbf{v}_A + \mathbf{\omega}_{AB} \mathbf{k} \times AB \mathbf{u}_{B/A} \]

\[ \mathbf{a}_B = \mathbf{a}_A + \mathbf{\alpha}_{AB} \mathbf{k} \times AB \mathbf{u}_{B/A} - (\mathbf{\omega}_{AB})^2 AB \mathbf{u}_{B/A} \]
Relative Motion: ref - Meriam & Kraige 2/13

Given:
- Two cars A & B at the instant shown
  \( v_A = 72 \hat{i} \text{ km/hr} \)
  \( a_A = 1.2 \hat{i} \text{ m/s}^2 \)
  \( v_B = 54 \hat{e}_t \text{ km/hr}, \text{ constant speed} \)

Find:
- (A) \( v_{B/A} =? \)
- (B) \( a_{B/A} =? \)

Solution:
- Convert to consistent units
  \[ \text{(km/hr)} \times \frac{1}{3.6} = \text{(m/s)} \Rightarrow \]
  \( v_A = 72(\text{km/hr}) = 20(\text{m/s}) \)
  \( v_B = 54(\text{km/hr}) = 15(\text{m/s}) \)

- Motion RELATIVE TO A of interest
- Two coordinate axes are used
  - Simplifies \( \mathbf{v} \) & \( \mathbf{a} \) definitions
  - Illustrates "coordinate conversion" for expressing answers "in terms of" a unified set.
  \( \hat{e}_n = \cos 30^\circ \hat{i} - \sin 30^\circ \hat{j} \)
  \( \hat{e}_t = \mathbf{k} \times \hat{e}_n = \cos 30^\circ \hat{j} + \sin 30^\circ \hat{i} \)
Relative Motion:  ref ~Meriam & Kraige 2/13

(A) Relative Velocity

\[ \mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A \]

\[ = 15 \mathbf{e}_t - 20 \mathbf{i} \quad (m/s) \]

\[ = 15(\sin 30^\circ \mathbf{i} + \cos 30^\circ \mathbf{j}) - 20 \mathbf{i} \quad (m/s) \]

\[ \mathbf{v}_{B/A} = -12.5 \mathbf{i} + 13.0 \mathbf{j} \quad (m/s) = 18 \quad (m/s) @ -46^\circ \]

- Velocity Polygon Approach (Graphical)

\[ \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \]

\[ \Rightarrow \mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A \]

\[ \mathbf{v}_A = 20 \text{ m/s} \]

\[ \mathbf{v}_B = 15 \text{ m/s} \]

\[ \mathbf{v}_{B/A} = 18 \text{ m/s} \]

\[ \mathbf{e}_n = \cos 30^\circ \mathbf{i} - \sin 30^\circ \mathbf{j} \]

\[ \mathbf{e}_t = k \times \mathbf{e}_n = \cos 30^\circ \mathbf{j} + \sin 30^\circ \mathbf{i} \]
Relative Motion: ref – Meriam & Kraige 2/13

(B) Relative Acceleration

\[ \mathbf{a}_A = 1.2 \mathbf{i} \quad (m/s^2) \]

\[ \mathbf{a}_B = v \mathbf{e}_t + \frac{v^2}{\rho} \mathbf{e}_n \quad (m/s^2) = \frac{(15 \ m/s)^2}{150 \ m} \mathbf{e}_n \]

\[ = 1.5 \mathbf{e}_n \quad (m/s^2) \]

\[ \mathbf{a}_{B/A} = \mathbf{a}_B - \mathbf{a}_A \]

\[ = 1.5 \mathbf{e}_n - 1.2 \mathbf{i} \quad (m/s^2) \]

\[ = 1.5(\cos 30° \mathbf{i} - \sin 30° \mathbf{j}) - 1.2 \mathbf{i} \quad (m/s^2) \]

\[ \mathbf{a}_{B/A} = 0.1 \mathbf{i} - 0.75 \mathbf{j} \quad (m/s^2) = 0.76 (m/s^2) \angle -82° \]

- Acceleration Polygon (Graphical)

\[ \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \]

\[ \Rightarrow \mathbf{a}_{B/A} = \mathbf{a}_B - \mathbf{a}_A \]
Given: A balloon at an altitude of 60 m is rising at steady rate of 4.5 m/s. A car passes below at constant speed of 72 kph.

Find: Relative rate of separation 1 second later:

\[ a_C = 0 \quad a_B = 0 \quad (m/s^2) \]

\[ v_C = 20\hat{i} \quad v_B = 4.5\hat{j} \quad (m/s) \]

\[ r_C = 20t\hat{i} \quad r_B = (60 + 4.5t)\hat{j} \quad (m) \]

\[ r_{B/C} = \sqrt{r_{B/C} \cdot r_{B/C}} \]

\[ r_{B/C}^2 = (-20t)^2 + (60 + 4.5t)^2 \]

\[ 2r_{B/C} \dot{r}_{B/C} = 2(-20t)(-20) + 2(60 + 4.5t)4.5 \]

Evaluate @ t = 1 & divide through by 2 \( r_{B/C} \)

\[ r_{B/C} = 690.25 / 67.52 = 10.22 (m / s) \]

Alternative Method (Vectors!)

\[ \Rightarrow \text{Find the radial component of} \quad v_{B/C} = \dot{r}_{B/C} = \dot{r}_{B/C}e_{r_{B/C}} + r_{B/C}\ddot{e}_{B/C} \]

\[ r_{B/C} = \frac{V_{B/C} \cdot \frac{r_{B/C}}{|r_{B/C}|}}{|r_{B/C}|} = \left( v_B - v_C \right) \cdot \frac{\dot{r}_B - \dot{r}_C}{|\dot{r}_B - \dot{r}_C|} = \left( -20, 4.5 \right) \cdot \frac{\left( -20, 64.5 \right)}{|67.5|} = \frac{690.3}{67.5} = 10.2 (m / s) \]
**Relative Motion**

\[ \mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \]
\[ \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \]
\[ \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \]

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**Special Case: Rigid Bodies**

When A & B are two points on the same rigid body:
- the relative motion is circular
- \( \mathbf{v}_{B/A} \) is perpendicular (\( \perp \)) to \( \mathbf{r}_{B/A} \)
  & \( |\mathbf{v}_{B/A}| = |\omega_{AB} AB| \)

\[ \mathbf{v}_B = \mathbf{v}_A + \omega_{AB} \mathbf{k} \times AB\mathbf{u}_{B/A} \]
\[ \mathbf{a}_B = \mathbf{a}_A + \alpha_{AB} \mathbf{k} \times AB\mathbf{u}_{B/A} - (\omega_{AB})^2 AB\mathbf{u}_{B/A} \]
Two points on a rigid body:

\[ \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \]

\[ \mathbf{v}_B \mathbf{i} = \mathbf{v}_A \mathbf{i} + \omega_{AB} \mathbf{k} \times \mathbf{AB} \mathbf{u}_{B/A} \]

\[ \mathbf{v}_B \mathbf{i} = \mathbf{v}_A \mathbf{i} - \mathbf{AB} \omega_{AB} (\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \]

Equating \( \mathbf{i} \) & \( \mathbf{j} \) components:

\[ \mathbf{i} \rightarrow \mathbf{v}_A - \mathbf{AB} \omega_{AB} \sin \theta = 0 \]

\[ \mathbf{j} \rightarrow \mathbf{v}_B = \mathbf{AB} \omega_{AB} \cos \theta \]

\[ \frac{\mathbf{v}_A}{\mathbf{v}_B} = \frac{\mathbf{AB} \omega_{AB} \sin \theta}{\mathbf{AB} \omega_{AB} \cos \theta} \]

\[ \frac{\mathbf{v}_A}{\mathbf{v}_B} = \frac{\sin \theta}{\cos \theta} \]
Instant Centers (velocity)

On every rigid body in general plane motion there exists a point \( \mathbf{P} \) where \( V_p = 0 \). It is known as instantaneous center (IC) of zero velocity or instantaneous center of rotation (ICR).

How to Locate IC?

1. Every point’s velocity vector is perpendicular to its relative position vector from the instant center.
2. Its speed (velocity magnitude) is proportional to its distance from IC.

\[
\omega = \pm \frac{V_A}{r_{IC-A}} = \frac{|V_B|}{r_{IC-B}} = \ldots
\]

At any instant, @ point of contact

If \( p_2 \) on the ground \( \Rightarrow V_{p_2} = 0 \)

\[
V_A = V_{p_1} + \omega \times r_{pA} = 0 + \omega \times r_{pA} \\
V_B = V_{p_1} + \omega \times r_{pB} = 0 + \omega \times r_{pB} \\
V_C = V_{p_1} + \omega \times r_{pC} = 0 + \omega \times r_{pC}
\]

\[
|V_A| = r_{pA} \omega \\
|V_B| = r_{pB} \omega \\
|V_C| = r_{pC} \omega
\]

⇒ \( V_{p_1}/p_2 = 0! \)

⇒ \( V_{p_1} = 0 \)

Rolling disk/tire (no slip!!)
**Knowing location of IC => Very useful tool!**

The direction of velocity for all points on the rigid body are known to be perpendicular to the line from IC to that point.

- If IC located and velocity of any one point is known:
  \[ \omega = \pm \frac{V_A}{r_{pA}} \]
  CW or CCW?

- If IC located and magnitude of \( \omega \) is known, the velocity of any point D is:
  \[ \mathbf{v}_D = \omega \times r_{pD} = \omega r_{pD} \mathbf{e}_{\perp pD} \]

**Special cases:**

Construction lines are parallel, not collinear
Mathematically the IC is at infinity!
*Pure Translation!*

\[ \mathbf{v}_A = \mathbf{v}_B \Rightarrow \omega = \pm \frac{|V_A|}{\infty} = 0 \]

The construction lines are collinear!
Speed is proportional to distance from IC.

\[ \omega = \pm \frac{|V_A|}{r_{pA}} = \frac{|V_B|}{r_{pB}} \]
Using Instant Centers (IC):

\[ V_A = AC \omega_{AB} \ [i] \]
\[ V_B = BC \omega_{AB} \ [-i] \]

\[ AC = AB \sin\theta \]
\[ BC = AB \cos\theta \]

\[ \frac{v_A}{v_B} = \frac{AB \omega_{AB} \sin\theta}{AB \omega_{AB} \cos\theta} \]
\[ \frac{v_A}{v_B} = \frac{\sin\theta}{\cos\theta} \]
Slider Crank Velocities Using Graphical & Instant Centers (IC):

\[ V_B = OB \quad \omega_o = CB \quad \omega_{AB} \]
\[ V_A = CA \quad \omega_{AB} \]

Be sure to account for direction!

\[ V_A = (OB / CB) \quad CA \quad \omega_o \]
**Example: Planar Kinematics of Rigid Bodies**  

*Meriam&Kraige Ex 5.8*

**Given:**
- Crank CB oscillates about C through a limited arc causing rocker OA to oscillate about O. When crank CB reaches horizontal, OA is vertical and the angular velocity of CB is 2 radians per second counterclockwise (CCW). For this instant,

**Find:**
A. The angular velocity of link AB  
B. The angular velocity of link OA

**Solution:**
- Three rigid bodies (links) need kinematics (velocities) to be established  
  - OA & CB pure rotation (1DOF each => $\omega_{OA}$ & $\omega_{CB}$)  
  - AB exhibits general plane motion (3 DOF)  
- Pin joints relate the kinematics (motion) of coincident points on the separate RB’s.
Example: Planar Kinematics of Rigid Bodies

Solution (cont’d):

- Relative velocity relationships for pairs of points on the three links

  (1) \( \mathbf{v}_B = \mathbf{v}_C + \mathbf{v}_{B/C} = \mathbf{0} + \omega_{CB} \times \mathbf{r}_{B/C} \)
  \( = (2 \text{ r/s}) \mathbf{k} \times (-75 \text{ mm}) \mathbf{i} \)
  \( = -150 \text{ mm/s j} \)

  (2) \( \mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{A/O} = \mathbf{0} + \omega_{OA} \times \mathbf{r}_{A/O} \)
  \( = (\omega_{OA} \text{ r/s}) \mathbf{k} \times (100 \text{ mm}) \mathbf{j} \)
  \( = -100 \omega_{OA} \text{ mm/s i} \)

  (3) \( \mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B} = \mathbf{v}_B + \omega_{AB} \times \mathbf{r}_{A/B} \)
  \( = -150 \text{ mm/s j} + (\omega_{AB} \text{ r/s}) \mathbf{k} \times \{(75 - 250 \text{ mm}) \mathbf{i} + (100 - 50 \text{ mm}) \mathbf{j}\} \)
  \( = (-175 \omega_{AB} - 150) \mathbf{j} - 50 \omega_{AB} \mathbf{i} \text{ (mm/s)} \)

- From (2) & (3), equating \( \mathbf{i} \) & \( \mathbf{j} \) components
  \( \mathbf{j} \Rightarrow 0 = (-175 \omega_{AB} - 150) \Rightarrow \omega_{AB} = -150/175 = -6/7 \text{ (r/s), i.e.CW} \)
  \( \mathbf{i} \Rightarrow -100 \omega_{OA} = -50 \omega_{AB} \Rightarrow \omega_{OA} = 50/100 \omega_{AB} = -3/7 \text{ (r/s), i.e.CW} \)
Example: Planar Kinematics of Rigid Bodies

Alternate: Graphical Solution (cont’d):

- Construct velocity polygon for the relative velocity constraint

\[ \mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B} \]

\[ \mathbf{r}_M = \mathbf{r}_O \mathbf{r}_M \]

- As before, \( \mathbf{v}_B \) easily computed

\[ (1) \quad \mathbf{v}_B = \omega_{CB} \mathbf{r}_{B/C} \perp \mathbf{r}_{B/C} \]

\[ = (2 \text{ r/s})(75 \text{ mm}) \mathbf{j} = -150 \text{ mm/s} \mathbf{j} \]

- \( \mathbf{v}_{A/B} \) is perpendicular (\( \perp \)) to \( \mathbf{r}_{A/B} \) &

\[ |\mathbf{v}_{A/B}| = \omega_{A/B} \mathbf{r}_{A/B} \]

- \( \mathbf{v}_A \) is horizontal (\( \perp \) to \( \mathbf{r}_{A/O} \))

\[ |\mathbf{v}_A| = \omega_{OA} \mathbf{r}_{A/O} \]

- Intersection of lines of action for \( \mathbf{v}_A \) & \( \mathbf{v}_{A/B} \) sets actual sizes for each vector

- Now measure (&/or compute) size of each vector based on scale used for \( \mathbf{v}_B \)
Example: Planar Kinematics of Rigid Bodies

Graphical Solution (cont’d):

- From the velocity polygon geometry $v_A$ and $v_{A/B}$ thus $\omega_{OA}$ and $\omega_{A/B}$ can be found

  \[ |v_A| = |v_B| \tan \theta = 150 \frac{50}{175} = 300/7 \text{ (mm/s)} \]

  \[ \Rightarrow \omega_{OA} = \pm \frac{|v_A|}{|r_{AO}|} = 300/7 \text{ (mm/s)} \frac{100}{100} \text{ (mm)} = 3/7 \text{ (r/s) CW} \]

  \[ |v_{A/B}| = |v_B|/\cos \theta = 150 \frac{150}{175} = 182 \times 6/7 \text{ (mm/s)} \]

  \[ \Rightarrow \omega_{A/B} = \pm \frac{|v_{A/B}|}{|r_{AB}|} = 182 \times 6/7 \text{ (mm/s)} \frac{182}{182} \text{ (mm)} = 6/7 \text{ (r/s) CW} \]

- Velocity polygon can be used to quickly validate your answers and/or determine rotation directions
Kinetics Summary

- Three general solution approaches for establishing the governing equations of motion (EOM) => Which one to use?
  
i) Newton’s Laws

\[ \sum F = ma_{CG} \quad \sum M_p = I_{CG} \alpha + r_{eff} ma_{CG} \]


\[ U_{A-B} = \int_{s_A}^{s_B} ma_t \, ds = \int_{v_A}^{v_B} mv \, dv = \frac{1}{2} m \left( v_B^2 - v_A^2 \right) = \Delta T_{A-B} \]

\[ U_{NC} = \Delta T + \Delta V_g + \Delta V_e = \Delta E_{TOT} \]

iii) Impulse - Momentum & Conservation of Momentum

- Typical forces

\[ I = \int F_R \, dt = \int dL = \Delta L \]

- Springs \( F = k (s - s_0) \)
- Friction \( F_f = \mu_{s/k} N \)
- Gravitation \( F = mg \)
Particle Kinetics: Free Body Diagrams

- Free Body Diagrams:
  - Isolate the particle/system of interest (i.e. boundaries)
  - For noting action-reaction between particles/bodies it is important to identify the common normal-tangent @ the point of contact (often one or the other is easily identified)

- Include ALL forces (& later => moments)
  - Field forces (gravity, electro-magnetic fields etc)
  - Viscous forces (aerodynamic drag, fluid flows, etc)
  - Contact forces (touching elements) -- Most common

- For motion over an interval --- draw in a general position!
Kinetics of Rigid Bodies – Newtons Law (2D)

\[ \sum \mathbf{F}_{\text{ext}} = m \mathbf{a}_{\text{CG}} \rightarrow 2 \text{ Kinetic constraints: } (x,y), (r,\theta), (n,t) \]

\[ \sum \mathbf{M}_{\text{CGz}} = \mathbf{I}_{\text{CGz}} \alpha \rightarrow +1 \text{ Rot. Kinetic constraint} \]

- 3 Kinetic Constraints per Rigid Body!

- Alternate Form: for \( \sum \mathbf{M}_P \) where \( P \neq \text{CG} \)

\[ \sum \mathbf{M}_P = \mathbf{I}_{\text{CG}} \alpha + (\mathbf{r}_{G/P} \times m \mathbf{a}_{\text{CG}})_z \]

\[ = \mathbf{I}_{\text{CG}} \alpha + m a_{\text{CG}} (\pm d_{\text{eff}}) \]

**OR**

\[ \sum \mathbf{M}_P = \mathbf{I}_P \alpha + (\mathbf{r}_{G/P} \times m \mathbf{a}_P)_z \]

→ IFF \( P \) is fixed, \( \mathbf{a}_P = 0! \)

\[ \sum \mathbf{M}_P = \mathbf{I}_P \alpha \]
Rigid Body Kinetics – Planar Motion (2D)

Given:
- A sliding warehouse door rides on ideal rollers & weighs 100#
- Assume the door weight is uniformly distributed

Find:
- The reactions at the roller supports
- The acceleration of the door.

Solution:
- Rectilinear motion: horizontal, no rotation
- IDEAL Rollers: Frictionless, massless
- Construct FBD with reactions properly AT POINT OF CONTACT!
Rigid Body Kinetics – Planar Motion (2D)

Solution (continued):

- Newton’s Law (3 kinetic constraints/RB)
  \[ \Sigma F_x : 20 \# = (100 \# / g) \ a_{CG_x} \]
  \[ a_{CG_x} = g / 5 \text{ ft/s}^2 \ (g = 32.2 \text{ ft/s}^2) \]

- \[ \Sigma F_y : R_A + R_B - 100 (\#) = m \ a_{CG_y} = 0 ! \]

- \[ \Sigma M_{CG} (CCW+) : 5 (R_B - R_A) - 2 (20) (ft-\#) = I_{CG} \alpha = 0 ! \]

- Use last two equations to resolve the two unknown reactions \( R_A \) & \( R_B \)
  \[ R_A = 46 \# \]
  \[ R_B = 54 (\#) \]
Rigid Body Kinetics – Planar Motion (2D)

Alternate Solution (continued):

- Newton’s Law (3 kinetic constraints/RB)
  \[ \Sigma F_x \Rightarrow a_{CG_x} = \frac{g}{5} \text{ ft/s}^2 \]

- Sum moments about a point other than CG
  \[ \Sigma M_P = I_{CG} \alpha + (r_{CG/P} \times ma_{CG})_z \quad \& \quad \alpha = 0! \]

\[ \Sigma M_A: \ 10 \ R_B + 1 \cdot 20 - 100 \cdot 5 \ (\text{ft-#}) = \left(\frac{100}{g}\right)\left(\frac{g}{5}\right)(3) \ (\text{slg-ft}^2/\text{s}^2) \]

  \[ \Rightarrow R_B = 54 \ (#) \]

\[ \Sigma M_B: -10 \ R_A + 1 \cdot 20 + 100 \cdot 5 \ (\text{ft-#}) = \left(\frac{100}{g}\right)\left(\frac{g}{5}\right)(3) \ (\text{slg-ft}^2/\text{s}^2) \]

  \[ \Rightarrow R_A = 46 \ # \]
Rigid Body Kinetics – Planar Motion (2D)

Given:
- A thin ring of mass \( m \) is free to rotate in the vertical plane about the frictionless pin joint at \( O \).
- Its angular velocity is \( \omega_0 \) (CW) when \( \theta = 0^\circ \)

Find: (for any arbitrary angle \( \theta \))
- The reactions forces at \( O \)
- The angular velocity of the ring

Solution:
- Fixed axis rotation about \( O \)
- Frictionless pin joint
- Construct FBD using \( n-t \) axes
  (+z into page – CW +)
Rigid Body Kinetics – Planar Motion (2D)

Solution (continued):

- Newton’s Law (3 kinetic constraints/RB)
  \[ \Sigma F_n : R_{On} - mg \sin \theta = m a_{Cn} \]
  \[ \Sigma F_t : R_{Ot} + mg \cos \theta = m a_{Ct} \]

- 3 kinetic constraints & 5 unknowns: \( R_{on}, R_{ot}, a_{Cn}, a_{Ct}, \alpha \)
- Look to **kinematics** to provide necessary constraints!
  - Fixed axis rotation \( \Rightarrow \) \( a_{Cn} = \omega^2 r \) \& \( a_{Ct} = \alpha r \)
- Now 3 kinetic + 2 kinematic constraints & 6 unknowns \((\omega)!\)

\[ \alpha \leq derivative \omega \quad 6 \text{ equations } \Leftrightarrow 6 \text{ unknowns C.B.S.}! \]
Rigid Body Kinetics – Planar Motion (2D)

Solution (continued):

- Combine $\Sigma M$ & $I_C$
  
  $$-R_{ot} \cdot r = I_C \alpha \Rightarrow \alpha = -\frac{R_{ot} \cdot r}{mr^2}$$
  
  $$\alpha = -\frac{R_{ot}}{mr}$$

- Combine $\Sigma F_t$ & $a_{ct} = \alpha \cdot r$
  
  $$R_{ot} + mg \cos \theta = m \alpha \cdot r$$

- Sub for $\alpha$ & resolve $R_{ot}$
  
  $$R_{ot} + mg \cos \theta = m \cdot r \left[-\frac{R_{ot}}{mr}\right] \Rightarrow R_{ot} = -\frac{mg \cos \theta}{2}$$

- Now $\alpha$ can be determined
  
  $$\alpha = -\left(-\frac{mg \cos \theta}{2}\right)/(mr) \Rightarrow \alpha = \frac{g \cos \theta}{2r}$$

- Remaining unknowns: $R_{on}$, $a_{cn}$, $\omega$ => Now what?
Rigid Body Kinetics – Planar Motion (2D)

Solution (continued):

- Knowing $\alpha = \frac{g \cos \theta}{2r}$
  
  o Integrate to get $\omega = f_2(\theta)$
  
  o Use $\omega$ to get $a_{Cn} = \omega^2 r$
  
  o Use $a_{Cn}$ & $\Sigma F_n$ to get $R_{On}$

- Variables ($\alpha$, $\omega$, $\theta$), no $t \Rightarrow$ use $\alpha d\theta = \omega d\omega$ form

$$\int_0^\theta \frac{g}{2r} \cos \theta \, d\theta = \int_{\omega_0}^{\omega} \omega \, d\omega$$

$$\Rightarrow \frac{g}{2r} \sin \theta \bigg|_0^\theta = \frac{1}{2} \omega^2 \bigg|_{\omega_0}^{\omega} \Rightarrow \omega^2 = \omega_0^2 + \frac{g}{r} \sin \theta$$

$$a_{Cn} = \omega_0^2 r = (\omega_0^2 + \frac{g}{r} \sin \theta) r = r \omega_0^2 + g \sin \theta$$

$$R_{Oa} - mg \sin \theta = m(r \omega_0^2 + g \sin \theta)$$

$$R_{On} = mr \omega_0^2 + 2mg \sin \theta$$

- Note $\Sigma M_O = I_O \alpha$ & eliminates reactions!

$$I_O = mr^2 + mr^2$$

$$mgr \cos \theta = 2mr^2 \alpha$$

$$\alpha = \frac{g \cos \theta}{(2r)}$$
Particle Kinetics: Path Coord Example  ref ~Meriam & Kraige 3/74

Given:
- The slider \((m=2 \text{ kg})\) fits loosely in the smooth slot of the disk which lies in a horizontal plane and rotates about a vertical axis through point \(O\).
- The slider is free to move only slightly along the slot in either direction before one (but not both) of the two wires \(#1\) or \(#2\) becomes taut.
- The disk starts from rest at time \(t = 0\) and has a constant clockwise angular acceleration of \(\alpha=0.5 \text{ r/s}^2\).

Find:
(A) Determine the TENSION \((T_2)\) in wire \(#2\) at \(t =1 \text{ second}\)
(B) Determine the REACTION FORCE \((N)\) between the slot and the block, again at \(t =1 \text{ second}\).
(C) Determine the TIME \((t)\) at which the tension in wire \(#2\) goes slack and wire \(#1\) becomes taut.

Solution:
- Asks for FORCES \((T,N)\) so we must first establish kinematics (accelerations!)
- “Move only slightly” means it is effectively fixed relative to the slot/disk, thus
- The slider travels a circle about \(O\) & path \((e_n,e_t)\) axes
  or polar \((e_r,e_\theta)\) axes are convenient
Solution (continued):

- Construct FBD
- Use disk kinematics \((\alpha=0.5 \text{ r/s}^2 \text{ CW constant})\) to determine slider’s total acceleration

\[
\rho = 0.100m = r \quad \Rightarrow \text{constant}
\]

\[
\therefore \rho = \rho = r = r = 0
\]

- Not instantaneous - integrate angular acceleration

\[
\int_0^\omega d\omega = \int_0^t \alpha \, dt = \int_0^t 0.5 \, dt
\]

\[
\omega = 0.5t
\]

\[
a_s = \alpha r \mathbf{e}_\theta - r \omega^2 \mathbf{e}_r
\]

\[
= v \mathbf{e}_t + \frac{v^2}{\rho} \mathbf{e}_n = \alpha r \mathbf{e}_t + \omega^2 r \mathbf{e}_n
\]
Particle Kinetics: Path Coord Example  

Solution (continued):

- Newton's Law can be applied along ANY two independent directions to resolve unknown reactions
  - Sum force components along \((n-t, r-\theta)\)
    
    \[ T_2 \cos 45 + N \sin 45 = m\alpha r \]
    
    \[ T_2 \sin 45 - N \cos 45 = -m\omega^2 r \]
  
  - OR to simplify algebra of unknowns, choose the directions along the unknown reactions and sum both forces and acceleration components
    
    \[ T = m \left( \alpha r \cos 45 - \omega^2 r \sin 45 \right) = \frac{mr\sqrt{2}}{2} (\alpha - \omega^2) \]
    
    \[ N = m \left( \alpha r \cos 45 + \omega^2 r \sin 45 \right) = \frac{mr\sqrt{2}}{2} (\alpha + \omega^2) \]

- **ASIDE:** This IS the geometric equivalent to simultaneously solving the first set of constraints to yield expressions for the unknowns
  - Noting the similarity of the expressions (± : + for \(N\), - for \(T\))
    
    \[ N, T_2 = \frac{mr\sqrt{2}}{2} (\alpha \pm \omega^2) \]
Particle Kinetics: Path Coord Example  ref ~Meriam & Kraige 3/74

Solution (continued):

- Substituting the known expressions for $\alpha$ & $\omega$

\[
N, T_2 = \frac{mr\sqrt{2}}{2} (\alpha \pm \omega^2)
= \frac{2 \times 0.1 \times \sqrt{2}}{2} \left\{ 0.5 \pm (0.5t)^2 \right\} (r/s^2)
\]

\[
N, T_2 = \frac{\sqrt{2}}{20} \left\{ 1 \pm 0.5t^2 \right\} (N)
\]

(A) So for $t=1$, the TENSION $T_2$ is

\[
T_2 = \frac{\sqrt{2}}{20} \left\{ 1 - 0.5(1)^2 \right\} (N) = \frac{\sqrt{2}}{40} (N) = 0.035 (N)
\]

(B) At $t=1$, the NORMAL REACTION $N$ is

\[
N = \frac{\sqrt{2}}{20} \left\{ 1 + 0.5(1)^2 \right\} (N) = \frac{3\sqrt{2}}{40} (N) = 0.106 (N)
\]

(C) The time when TENSION $T_2$ goes to zero is

\[
T_2 = \frac{\sqrt{2}}{20} \left\{ 1 - 0.5t^2 \right\} (N) = 0 \Rightarrow 1 - 0.5t^2 = 0 \Rightarrow t = \sqrt{2} \Rightarrow t = 1.414 (s)
\]
Particle Kinetics: Path Coord Example ref ~Meriam & Kraige 3/74

Langiappe:

- The acceleration vector starts off completely in the lateral (θ or t) direction here (ω=0). Since cables/wires/ropes cannot PUSH, only \( T_2 \) can be engaged in balancing the (\( r \) or \( n \)) component of the side wall reaction \( N \).
- The tangential acceleration component remains constant.
- As the disk speeds up (\( ω > 0 \)), the normal component increases.
- When the total acceleration vector aligns with the normal reaction force between the block & slot, the cord/wire tensions are both zero momentarily, and as \( T_2 \) goes slack, \( T_1 \) will become taut.
Kinetics of Rigid Bodies (2D): Impulse-Momentum

- Motion studies: **Forces/Moments, Velocities (linear/angular), Time**
  
  o **Linear Momentum**  \((\text{Vector constraint 2D})\)
    \[
    \mathbf{I}_{\text{ext}} = \int \mathbf{F}_{\text{ext}} \, dt = \Delta m \mathbf{v}_{\text{CG}} = \Delta \mathbf{L}_{\text{CG}}
    \]
  
  o **Angular Momentum**  \((+1 \text{ constraint})\)
    Add RB ROTATION to Moment of \(\mathbf{L}_{\text{CG}}\)
    \[
    AI_{P} = \int \mathbf{M}_{P} \, dt
    = I_{CG} \Delta \omega + (r_{G/P} \times m \Delta \mathbf{v}_{CG})_{z}
    = I_{P} \Delta \omega + (r_{G/P} \times m \Delta \mathbf{v}_{P})_{z}
    \]
    \(\rightarrow\) If \(P\) is CG or a fixed point in space
    \[
    AI_{CG_{z}} = \int \mathbf{M}_{CG_{z}} \, dt = I_{CG_{z}} \Delta \omega = \Delta H_{CG}
    \]
    \[
    AI_{P_{z}} = \int \mathbf{M}_{P_{z}} \, dt = I_{P_{z}} \Delta \omega = \Delta H_{P}
    \]

**Impact:** Coefficient of Restitution

Complicated phenomenon with limited applicability

\[
e = \frac{\left( V_{\text{rel-Sep}} \right)}{\left( V_{\text{rel-App}} \right)}
\]

\(H_{P} = I_{CG} \omega + r_{G/P} \times m \mathbf{v}_{CG}\)

\(H_{P} = I_{CG} \omega + m v_{CG} d_{eff}\)
Example: Conservation of Momentum

Given:
- An artillery gun \((m_G)\) resting on the ground, fires a shell \((m_P)\) with a speed \(v_p\)

Find:
(A) The recoil speed \((v_R)\) of the gun

Solution:
- Rectilinear motion (i.e. only horizontal motion of interest here)
- FBD of system components, up through the shell leaving the gun barrel
- Propellant firing is internal to the system
  - System momentum is conserved in the horizontal direction

\[
\Delta L_{\text{sys-x}} = 0
\]

\[
\Delta L_{\text{sys-x}} = m_G(v_G - 0) + m_P(v_P - 0) = 0
\]

\[
v_R = -v_G = \frac{m_P}{m_G} v_p
\]
Example: Conservation of Momentum

Given:
- More often, a “muzzle velocity” \( (v_{P/G}) \) or speed of the shell relative to the gun barrel is specified

Find:
(A) The recoil speed \( (v_R) \) of the gun

Solution:
- FBD (same), rectilinear motion (same) & propellant firing is internal (same)

\[
\Delta L_{sys} = 0
\]
\[
\Delta L_{sys-x} = m_G (v_G - 0) + m_P (v_P - 0) = 0
\]
\[
v_P = v_G + v_{P/G}
\]
\[
m_G v_G + m_P (v_G + v_{P/G}) = 0
\]
\[
v_R = -v_G = \left( \frac{m_P}{m_G + m_P} \right) v_{P/G}
\]
Example: continued

Asking for more:
- If resultant muzzle blast occurs over a short time $t_{\text{blast}}$, what resultant "kick" is felt by the cannon?

Solution:
- An average $F_{\text{prop-avg}}$ can be computed to approximate the kick.

$$\Delta L_{\text{sys}} = 0 \quad \Delta L_{\text{gun-x}} \neq 0$$

$$\Delta L_{\text{gun-x}} = I_x = \int -F_{\text{propellant}} \, dt$$

$$= -F_{\text{prop-avg}} \int_0^{t_{\text{blast}}} \, dt = -F_{\text{prop-avg}} \, t_{\text{blast}}$$

$$F_{\text{prop-avg}} = \frac{-1}{t_{\text{blast}}} \Delta L_{\text{gun-x}} = \frac{-m_G}{t_{\text{blast}}} (-v_R - 0) = m_G \frac{v_R}{t_{\text{blast}}}$$

$$F_{\text{prop-avg}} = m_G \left( \frac{m_P}{m_G} \right) \frac{v_p}{t_{\text{blast}}} = m_P \frac{v_p}{t_{\text{blast}}} \quad \text{or} \quad F_{\text{prop-avg}} = m_G \left( \frac{m_P}{m_G + m_P} \right) \frac{v_{p/G}}{t_{\text{blast}}} = \left( \frac{m_G m_P}{m_G + m_P} \right) \frac{v_{p/G}}{t_{\text{blast}}}.\]
Example: Conservation of Momentum

Given:

- Numerous examples with similar circumstances, rephrasing the wording
  - Kid(s) on a boat in still water, one jumps off
  - Car lands on a barge & skids to rest relative to barge
  - Rail cars collide & stay attached

Find:

(A) The resulting speeds of each element
(B) A time it takes to “skid to rest”

Solution:

- Similar conservation of momentum relations

\[ \Delta L_{\text{sys-x}} = 0 \implies \text{Resolve velocities} \]
\[ \Delta L_{\text{components-x}} \neq 0 \implies \text{Velocities known} \implies \text{Resolve Net Impulse} \]

\[ I = \int F_R \, dt = \int dL = \Delta L \implies I = \bar{F}_{R-\text{avg}} \Delta t = \Delta L \]
Particle Kinetics: Impulse-Momentum

- **Impact Problems:**
  - Reformulation of one type of Impulse-Momentum \( I = \Delta L = m\Delta v \)
  - Impact Forces (\( F \)) characterized by
    - LARGE MAGNITUDE
    - SHORT TIME DURATION
    - Ex: explosions, collisions, ball-bat, club-golf ball
  - Neglect other conventional forces of lesser effect for the short time interval considered as their total effect is negligible
    - Springs
    - Gravity
    - Many Reaction forces (BUT NOT ALL!)
  - Good opportunity to look at the SYSTEM of particles in simplifying the problem (reactions are internal!)
Particle Kinetics: Impulse-Momentum/Impact

- Impact
  - Locate Common Normal/Tangent
    - Line of contact/impact - the NORMAL!
  - Forces (F) of interaction
    - Equal, Opposite, Co-linear
  - Very complex internal phenomena, captured by Coefficient of Restitution
    \[ e = \frac{\left( V_{\text{Relative-Separation}} \right)_{\text{Common Normal}}}{\left( V_{\text{Relative-Approach}} \right)_{\text{Common Normal}}} \]
    (good derivation in text --- READ IT!)

- Central & Oblique Impacts
  - **Central**: Velocities CO LINEAR with the line of impact (i.e. the common normal)
  - **Oblique**: Velocities are NOT co-linear
Particle Kinetics: Impulse-Momentum/Impact

- Solving Impact Problems!

1. Tangential direction: individual particles have no net external impulsive forces! $\mu=0$

   \[ m_A v_{At} = m_A v_{At}^* \quad \& \quad m_B v_{Bt} = m_B v_{Bt}^* \]

2. Normal direction: system of particles has no net external impulsive forces!

   \[ \Delta L_{SYS} \bigg|_n = 0 \Rightarrow m_A v_{An} + m_B v_{Bn} = m_A v_{An}^* + m_B v_{Bn}^* \]

3. Coefficient of Restitution: Rel. Velocities along Common NORMAL!

   \[ e = \frac{v_{Relative\ Separation}}{v_{Relative\ Approach}} \bigg|_{Normal} = \frac{v_{Bn}^* - v_{An}^*}{v_{An} - v_{Bn}} \]

   (Perfectly Plastic) $0 \leq e \leq 1$ (Perfectly Elastic)
Particle Kinetics: Impulse-Momentum/ Impact

- Solving constraint relations!

\( (1) \quad v_{At} = v_{At}^* \quad \& \quad v_{Bt} = v_{Bt}^* \)

\( (2) \quad v_{Bn}^* = v_{Bn} + \frac{m_A}{m_B} (v_{An} - v_{An}^*) \)

\( (3) \quad v_{Bn}^* = e (v_{An} - v_{Bn}) + v_{An}^* \)

- From which the unknown rebound (normal) component of velocities become

\( (2) \Rightarrow (3) \Rightarrow (4) \quad v_{An}^* = \left( \frac{m_A - m_B e}{m_A + m_B} \right) v_{An} + \left( \frac{m_B}{m_A + m_B} \right) (1+e) v_{Bn} \)

\( (4) \Rightarrow (3) \Rightarrow (5) \quad v_{Bn}^* = \left( \frac{m_A}{m_A + m_B} \right) (1+e) v_{An} + \left( \frac{m_B - m_A e}{m_A + m_B} \right) v_{Bn} \)

\[ \mathbf{\overline{V}_A} = (v_{A_t}, v_{A_n}^*) \quad \& \quad \mathbf{\overline{V}_B} = (v_{B_t}, v_{B_n}^*) \]
Particle Kinetics: Impulse-Momentum/Impact

- What if \( m_B \gg m_A \)?

\[
\begin{align*}
(1) \quad v_{At}^* &= v_{At}^* \quad \& \quad v_{Bt}^* = v_{Bt}^* \\
(2) \quad v_{Bn}^* &= v_{Bn} + \frac{m_A}{m_B}(v_{An} - v_{An}^*) \\
(3) \quad v_{Bn}^* &= e(v_{An} - v_{Bn}) + v_{An}^*
\end{align*}
\]

- From which the unknown rebound (normal) component of velocities become

\[
\begin{align*}
(2) &\rightarrow (4) \quad v_{Bn}^* = v_{Bn} \\
(4) &\rightarrow (3) \Rightarrow (5) \quad v_{An}^* = -e v_{An} + (1 + e)v_{Bn}
\end{align*}
\]

\[
\textbf{V}_A^* = (v_{At}^*, v_{An}^*) \quad \& \quad \textbf{V}_B^* = \textbf{V}_B
\]
Particle Kinetics: **WORK-ENERGY** for Rigid Bodies (Scalar!)

\[ U_{NC} = \Delta T + \Delta V_g + \Delta V_e = \Delta E_{TOT} \]

- Need to incorporate the **ROTATION** elements
  - **Kinetic Energy of Rigid Bodies:**
    \[ T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2 \]
    - For fixed axis of rotation \( P \) other than \( CG \).
      \[ v_G = r_{G/O} \omega^2 \]
      \[ T = \frac{1}{2} m (r_{G/O} \omega)^2 + \frac{1}{2} I_G \omega^2 \]
      \[ = \frac{1}{2} (I_G + m r_{G/O}^2) \omega^2 = \frac{1}{2} I_P \omega^2 \]
    - Use either \( CG \) or fixed axis of rotation \( P \)!!!
Particle Kinetics: WORK-ENERGY for Rigid Bodies (Scalar!)

\[ U_{NC} = \Delta T + \Delta V_g + \Delta V_e = \Delta E_{TOT} \]

- Need to incorporate the **ROTATION** elements
  - **Conservative Forces (now Moments):**
    - Springs (linear & torsional)
      \[ F_S = - k_s (l - l_0) \quad k_s \text{ – stiffness (Force/Length)} \quad l_0 \text{ – unstretched length} \]
      \[ M_S = - k_\theta (\theta - \theta_0) \quad k_\theta \text{ – torsional stiffness (torque/radian)} \quad \theta_0 \text{ – unstretched angle} \]
    - Potential Functions
      \[ \Delta V_e = \Delta V_{e_s} + \Delta V_{e_\theta} = \frac{1}{2} k_s (\Delta l_f^2 - \Delta l_i^2) + \frac{1}{2} k_\theta (\Delta \theta_f^2 - \Delta \theta_i^2) \]
    - Constant Torques can also be treated as Potential functions
      \[ \Delta V_{e_\theta} = M \Delta \theta \quad \Delta V_{g_y} = W \Delta h = mg \Delta h \]
Particle Kinetics: WORK-ENERGY for Rigid Bodies (Scalar!)

\[ U_{NC} = \Delta T + \Delta V_g + \Delta V_e = \Delta E_{TOT} \]

- Need to incorporate the **ROTATION** elements

  - **Work:**
    - **FORCE/MOMENT** applied thru a **CURVILINEAR/ANGULAR DISPLACEMENT**
      - No displacement -- NO WORK!
      
      \[ U_{1-2} = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r} + \int_{\theta_1}^{\theta_2} \mathbf{M} \cdot d\theta = \int_{s_1}^{s_2} F_t \, ds + \int_{\theta_1}^{\theta_2} M \, d\theta \]
      - Units ENERGY:  SI: Joules (1 N-m)  FPS: (lb-ft)

  - **Power**: work/time
    
    \[ P = \frac{dU}{dt} = \mathbf{F} \cdot \mathbf{v} + \mathbf{M} \cdot \omega \]
    - Units  SI: Watt (Joules/sec)  FPS: 1 Horsepower = 550 ft-lb/sec
Rigid Body Kinetics – Planar Motion (2D)

Revisit from last class:
- A thin ring of mass $m$ is free to rotate in the vertical plane about the frictionless pin joint at $O$.
- Its angular velocity is $\omega_0$ (CW) when $\theta=0^\circ$

**Find:** (for any arbitrary angle $\theta$)
- The angular velocity of the ring
- The reactions forces at $O$

**Solution:**
- Fixed axis rotation at frictionless pin joint $O$
- FBD constructed using $n-t$ axes (+$z$ into page – CW +)
- Forces, displacements, velocities => W-E!
  - *Last time Integrated $\Sigma M$ to get $\omega = f(\theta)$*

\[
U_{NC} = \Delta T + \Delta V_g + \Delta V_e = \Delta E_{TOT}
\]

$\Rightarrow \Delta E_{TOT} = 0$ !!!!
Rigid Body Kinetics – Planar Motion (2D)

W-E Solution (continued):

\[ \Delta E_{TOT} = \Delta T + \Delta V_g = 0 \]

\[ \Delta T = \frac{1}{2} I_o (\omega^2 - \omega_0^2) \]

\[ \Delta V_g = W \Delta h = mg \Delta h = mg(-r \sin \theta) \]

\[ \Delta E_{TOT} = \frac{1}{2} I_o (\omega^2 - \omega_0^2) - mgr \sin \theta = 0 \]

\[ \omega^2 = \omega_0^2 + \frac{2mgr \sin \theta}{I_o} = \omega_0^2 + \frac{2mgr \sin \theta}{2mr^2} \]

\[ \omega_\theta = \sqrt{\omega_0^2 + \frac{g}{r} \sin \theta} \]

- Reaction forces?  See earlier example using Newton's Laws
Conservation-Energy Example  ref Bedford & Fowler 15.85

Given:
- A small pellet of mass $m$ and negligible diameter, sits atop a smooth circular cylinder of radius $R$.
- The pellet is given a slight nudge

Find:
(A) Draw a correct FBD for the pellet in general position $\theta$
(B) The value of $\theta$ where pellet loses contact with the cylinder
(C) The pellet’s speed at the point where it loses contact

Solution:
- FBD of pellet in general position (working over a motion interval here)
- Identify
  - Conservative Forces $mg$ (Weight/Gravity)
  - Non-working Constraint Forces $N$ (Cylinder reaction force)
Conservation-Energy Example  ref Bedford & Fowler 15.85

Solution:

- ALL Forces are either Conservative or Non-working constraints, therefore Cons. Of Energy applies!
  \[ \Delta E_{sys} = \Delta T + \Delta V_g = 0 \]

- It starts from rest, \( v_\theta = 0 \) @ \( \theta = 0 \)
- Set the datum for potential @ base of the cylinder (y=Rcos\( \theta \))
  \[ \Delta E_{sys} = 0 = \Delta T + \Delta V_g \]
  \[ 0 = \frac{1}{2} m \left( v_\theta^2 - 0 \right) + mg \left( R \cos \theta - R \right) \]
  \[ v_\theta^2 = 2gR(1 - \cos \theta) \]

- Just as the pellet loses contact (N=0)

  \[ \sum F_n = ma_n \]
  \[ mg \cos \theta = m \frac{v_\theta^2}{R} \]
  \[ v_\theta^2 = Rg \cos \theta \]

Equating expressions for \( v_\theta^2 \) yields

\[ v_\theta^2 \Rightarrow Rg \cos \theta = 2gR(1 - \cos \theta) \]

\[ 3 \cos \theta = 2 \]

\[ \theta = \cos^{-1}(2/3) \approx 48^\circ \]
Rigid Body Kinetics – Planar Motion (2D)

Given:
- A rotating sheave \( (m_{50}) \) carries a high strength, electromagnet \( (m_{100}) \)
  \[ r = 0.4 \text{ m} \quad k_0 = 0.3 \text{ m} \quad m_{50} = 50 \text{ kg} \quad m_{100} = 100 \text{ kg} \]
- Released from rest with the spring initially stretched 0.1 m \( k_{spr} = 1.5 \text{ kN/m} \)

Find:
- Velocity of \( O \) after it has dropped \( \Delta y_O = 0.5 \text{ m} \)

Solution:
- 2 RB, CG’s motion rectilinear + sheave rotation
- Set coordinate \( x-y \) axes horiz/vert with CCW+
- BC’s loads, displacements, velocities \( \Rightarrow \) W-E!

\[ U_{NC} = \Delta T + \Delta V_g + \Delta V_e = \Delta E_{TOT} \]
- Finish FBD & see if system is conservative!
Rigid Body Kinetics – Planar Motion (2D)

W-E Solution (continued):

- Conservative loads: \( F_{spr}, m_{50}g, m_{100}g \)
- Forces DO NOT WORK: \( T \Rightarrow \text{displacement} = 0! \)
- \( R_{Oy} \) Internal reaction not requested – System?

\[
\Delta E_{sys} = \Delta T + \Delta V_g + \Delta V_e = 0
\]

\[
\Delta T_{sys} = \frac{1}{2}m_{100}(v_O^2 - 0) + \frac{1}{2}m_{50}(v_O^2 - 0) + \frac{1}{2}I_O(\omega^2 - 0)
\]

\[
= \frac{1}{2}(m_{100} + m_{50})v_O^2 + \frac{1}{2}m_{50}k_O^2\omega^2
\]

\[
\Delta V_g = W\Delta h = (m_{50} + m_{100})g\Delta y_O
\]

\[
\Delta V_e = \frac{1}{2}k_{spr}\{(\Delta S + S_O)^2 - S_O^2\}
\]

\[
\mathbf{V}_O = \mathbf{V}_C + \omega \mathbf{k} \times \mathbf{r}_I \Rightarrow v_O = \omega r
\]

\[
\Rightarrow v_S = \omega(2r) = 2v_O
\]
Rigid Body Kinetics – Planar Motion (2D)

W-E Solution (continued):

- Assembling the terms \( \Delta E_{sys} = \Delta T + \Delta V_g + \Delta V_e = 0 \)

\[
\Delta E_{TOT} = 0 = \left[ \frac{1}{2} \left\{ m_{100}r^2 + m_{50}(r^2 + k_O^2) \right\} \omega^2 \right. \\
+ \left( m_{50} + m_{100} \right) g \Delta y_O + \frac{1}{2} k_{spr} \left\{ (2 \Delta y_O + S_O)^2 - S_0^2 \right\} 
\]

\[
\omega = \sqrt{\frac{2 \left( m_{50} + m_{100} \right) g \Delta y_O + k_{spr} \left\{ (2 \Delta y_O + S_O)^2 - S_0^2 \right\}}{m_{100}r^2 + m_{50}(r^2 + k_O^2)}} 
\]

\[
r = 0.4 m \quad k_{spr} = 1.5 \text{ kN/m} \quad m_{100} = 100 \text{ kg} \\
\Delta y = -0.1 m \quad k_O = 0.3 m \quad S_0 = 0.1 m \quad m_{50} = 50 \text{ kg} 
\]

\[
\omega = 3.5r / s \quad \text{CW} 
\]

\[
\Rightarrow v_O = r \omega = 0.4m * 3.5r / s = 14 \text{ m/s} \quad (-j) 
\]
Conservation-Energy Example  ref ~Meriam & Kraige 3/17

Given:
- \( m = 3 \text{ kg slider on circular track shown} \)
- Starting from A with \( v_A = 0 \)
- \( l_o = 0.6 \text{ m (unstretched), } k = 350 \text{ N/m} \)
- \( \mu = 0 \) (i.e friction is negligible)

Find:
(A) Velocity of slider as it passes B

Solution:
- FBD of crate in general position
  (working over a motion interval here)
- Identify
  - Conservative Forces
    \( mg \) (Weight/Gravity) & \( F_s \) (Spring)
  - Non-working Constraint Forces
    \( N \) (Track reaction force)
Conservation-Energy Example

Solution:

- ALL Forces are either Conservative or Non-working constraints, therefore Cons. Of Energy applies!

\[ \Delta E_{TOT} = \Delta T + \Delta V_g + \Delta V_e = 0 \]

\[ \Delta T_{AB} = \frac{1}{2} m (v_B^2 - v_A^2) = \frac{1}{2} m (v_B^2 - 0) \]

\[ \Delta V_{ABg} = mg (y_B - y_A) = mg (0 - R) \]

\[ \Delta V_{ABe} = \frac{1}{2} k \left\{ (l_B - l_0)^2 - (l_A - l_0)^2 \right\} \]

- Pulling together all components & isolating \( v_B \)

\[ v_B = \sqrt{2gR + \frac{k}{m} \left\{ R^2 - (\sqrt{2}R - R)^2 \right\}} \]

- Incorporating numerical values of all terms

\[ v_B = \sqrt{2 \times 9.81 \text{ (m/s}^2)(0.6m) + \frac{350 \text{ N/m}}{3 \text{ kg}} \left\{ (0.6m)^2 - (\sqrt{2} \times 0.6m - 0.6m)^2 \right\}} = 6.82 \text{ m/s} \]
Work-Energy Example  ref ~Meriam & Kraige 3/11

Given:
- A crate of mass $m$ slides down an incline
- $m=50 \text{ kg}$, $\theta=15^\circ$, $\mu_k=0.3$,
- Reaches A with speed $4 \text{ m/s}$

Find:
(A) Speed of crate $v_B$ as it reaches a point B 10 m down the incline from A

Solution:
- Rectilinear motion, align axes accordingly -i.e. $\parallel$ & $\perp$ to incline
- FBD of crate in general position (working over a motion interval here)
- No movement $\perp$ to incline so Newton's Law says -?

$$\sum F_y = mg \cos \theta - N = 0 \quad \Rightarrow \quad N = mg \cos \theta$$
Work-Energy Example

Solution (cont’d):

- Work done is due to the resultant forces in direction of displacement (i.e. down incline) & includes Friction & component of Weight

\[
U_{A-B} = (mg \sin\theta - N\mu_k)\Delta x_{AB} = (mg \sin\theta - mg \cos\theta\mu_k)\Delta x_{AB}
\]

- Principle of Work-Energy then says

\[
U_{A-B} = \Delta T_{A-B} = T_B - T_A
\]

\[\Rightarrow T_B = U_{A-B} + T_A\]

\[
\frac{1}{2}mv_B^2 = mg(\sin\theta - \cos\theta\mu_k)\Delta x_{AB} + \frac{1}{2}mv_A^2
\]

\[v_B = \sqrt{2g(\sin\theta - \cos\theta\mu_k)\Delta x_{AB} + v_A^2}\]

\[v_B = \sqrt{2 \times 9.81(m/s^2) \times (\sin15^\circ - \cos15^\circ \times 0.3) \times 10m + (4m/s)^2}\]

\[v_B = 3.15 \text{ m/s}\]
Work-Energy: Example ref ~Meriam & Kraige 3/13

Given:
- Block \( (m = 50 \text{ kg}) \) mounted on rollers
- Massless spring w/ \( k = 80 \text{ N/m} \)
- Released from rest at A where spring has initial stretch of 0.233 m
- Cord w/ constant tension \( P = 300 \text{ N} \) attaches to block & routed over frictionless/massless (ideal) pulley @ C

Find:
(A) Speed of block \( v_B \) as it reaches a point B directly under the pulley.

Solution:
- Again, rectilinear motion, align axes accordingly
- FBD of block in general position (working over a motion interval here)
- Look at alternative - include the rope in as part of the SYSTEM - reduce FDB to an **ACTIVE Force Diagram!**
Solution (cont’d):

**ACTIVE Force Diagram!**

- Eliminate Normal Forces \( \perp \) to displacement @ their point of contact {THEY DO NO WORK!}
  - Weight \((mg)\) & Roller reactions \((N)\)
  - Pulley force on rope \((R)\)

- Active forces DO work on the system
  - Spring Force \((F_s)\) => opposes motion

\[
F_s = -kx
\]

\[
U_{ABf_s} = \int_{x_A}^{x_B} F_s \, dx = \int_{x_A}^{x_B} -kx \, dx
\]

\[
= -\frac{1}{2} kx^2 \bigg|_{x_A}^{x_B} = -\frac{1}{2} k(x_B^2 - x_A^2)
\]

- Assuming block can actually reach B

\[
U_{ABf_s} = -\frac{1}{2} 80(N/m) \{ (1.2 + 0.233)^2 - 0.233^2 \} (m^2) = -80 \text{Joules}
\]
Work-Energy: Example

Solution (cont’d):

- Calculate Work done on system by $P$
  - Cord Tension ($P$) $\Rightarrow$ constant
  - Displacement of $P$

$$L_{cord} = s_P + l = constant$$

$$\Delta s_P = -\Delta l = l_A - l_B$$

$$= \sqrt{1.2^2 + 0.9^2} - 0.9 \approx 0.61 m$$

$$U_{ABp} = P\Delta s = 300(N) \times 0.61(m)$$

$$= 180 \text{ Joules}$$

- Work-Energy

$$U_{TOT} = \Delta T_{A-B} = T_B - T_A$$

$$-80 + 180(\text{Joules}) = \frac{1}{2} m(v_B^2 - 0)$$

$$\Rightarrow v_B = \sqrt{\frac{100(\text{Joules}) \times 2}{50 Kg}} = 2.0 \text{ m/s}$$

Active Force Diagram
Given: $\omega_c = 2 \text{ r/s}$
$\alpha_c = 6 \text{ r/s}^2$

Find: $v_D, a_D$
\[ r_A = 6'' \quad r_B = 12'' \quad r_C = 8'' \]
\[ \omega_C = 2 \frac{r}{s} \quad \alpha_C = 6 \frac{r}{s^2} \]
\[ V_{E1} = r_C \omega_C = 8 \cdot 2 = 16 \text{ in/s} \uparrow \]
\[ V_{E2} = 16 \uparrow = r_B \omega_B = 12 \omega_B \]
so: \[ \omega_B = \frac{4}{3} = \omega_A \quad [r/s] \]
\[ V_D = V_F = \omega_A r_A = \frac{4}{3} (6) = 8 \uparrow \quad [\text{in/s}] \]

\[ a_{E1} = \alpha_C r_C \uparrow + \omega_C^2 r_C \rightarrow \]
\[ = 6 (8) \uparrow + 4 (8) \rightarrow = 48 \uparrow + 32 \rightarrow \quad [\text{in/s}^2] \]
\[ a_{E2t} = a_{E1t} = 48 \uparrow = \alpha_B r_B = \alpha_B (12) \quad \alpha_B = 4 \quad \alpha_B = \alpha_A \]
\[ a_{Ft} = \alpha_A r_A = 4 (6) = 24 \uparrow = a_D \quad [\text{in/s}^2] \]
Given: \( r_o = 3' \), \( r_i = 2' \), \( v_o = 10 \text{ f/s} \), no slip

Find: \( v_B \)

\[ v_o = 10 \text{ ft/s} \rightarrow \]

\[ v_c = v_o + \omega r = 10 - 2 \omega = 0 \]

\( \omega = 5 \text{ } \omega \text{ or } -5 \text{ } k \)

\[ v_B = v_o + \omega \times r_{B/o} = 10 \textbf{i} + (-5 \textbf{k}) \times -3 \textbf{i} = -5 \textbf{i} \text{ [ft/s]} \]

or

\[ v_B = v_c + \omega \times r_{B/c} = 0 + 5 \textbf{k} \times -1 \textbf{i} = -5 \textbf{i} \text{ [ft/s]} \]