Dynamics Problem Decomposition

Dynamics

Kinematics

Geometric descriptions of motion & constraints

Kinetics

Loading relationships which dictate CHANGES in motion
Dynamic Studies

Dynamics: (Kinematics & Kinetics)

Particles

Rigid Bodies

- \( M \) - mass
- \((x,y)\) - position in 2D
- 2 Degrees of Freedom (DOF) in the plane

- \( M \) & \( I \) - both mass & rotational inertia
- \((x,y,\theta)\) - position & orientation in 2D (3DOF)

Getting Started => Particle Kinematics

- Rectilinear Motion
  - Movement along a straight line in 1-2 or 3D
    - 1 Degree of Freedom (DOF)* - \( s(t) \)

- Curvilinear Motion
  - Movement of particle along an arbitrary path through space
Rectilinear Motion Overview (Calculus/Physics Review!):

- **Position** - $s(t)$
- **Speed** - $v(t)$
  
  (1) $v = \frac{ds}{dt} = s$
- **Acceleration** - $a(t)$
  
  (2) $a = \frac{dv}{dt} = v = \frac{d^2 s}{dt^2} = s$

**Basic Calculus!**
- Max’s & Min’s ?
- Undo Differentiation?

**Typical Functions ??**
- Polynomials
- Trigonometric
- Logarithms
- Exponentials

Rectilinear Motion Summary:

- **Position** - $s(t)$
- **Speed** - $v(t)$
  
  (1) $v = \frac{ds}{dt} = s$
- **Acceleration** - $a(t)$
  
  (2) $a = \frac{dv}{dt} = v = \frac{d^2 s}{dt^2} = s$

(2*) $v \ dv = a \ ds$

$a(t)$ => Solid Rocket Propulsion  
$a(v)$ => aerodynamic drag

$a(s)$ => Gravitational fields, springs, conservative forces etc.
**Rectilinear Kinematics: Accel. a function of velocity – a(v)**

Given:
- A freighter moving at 8 knots when engines are stopped
- Deceleration $a = -kv^2$
- Speed reduces to 4 knots after ten minutes

Find:
(A) Speed of the ship as a function of time $v(t)$
(B) How far does the ship travel in the 10 minutes it takes to reduce the speed by 1/2?

Solution:
(A) With $a, v$ & $t$ parameters given/requested, use $a=\frac{dv}{dt}$ form

$$ t_f = t(v) = \int_{v_i}^{v_f} \frac{dv}{a(v)} - t_i = \int_{v_i}^{v_f} \frac{dv}{-kv^2} + 0 $$

$$ \Rightarrow t_f = \frac{1}{kv}|_{v_i}^{v_f} = \frac{1}{k} \left( \frac{1}{v_f} - \frac{1}{v_i} \right) \quad \Rightarrow v_f = v(t_f) = \frac{8}{8kt_f + 1} \text{ (knots)} $$
and the resulting expression for speed of the ship as a function of time \( v(t) \) is as follows
\[
v(t) = \frac{8}{6t + 1} \text{ (knots)}
\]

From here, there are two alternatives for resolving the second question

**Rectilinear Kinematics: Accel. a function of velocity – \( a(v) \)**

**METHOD 1:** Now, knowing the velocity as a function of time \( v(t) \), the boat’s position can be found by integration
\[
\int_0^t ds = \int_0^t \frac{8}{6t + 1} dt
\]
\[
s_f - 0 = \left. \frac{4}{3} \ln(6t + 1) \right|_0^t = \frac{4}{3} \left( \ln(6t + 1) - \ln(1) \right)
\]
and the resulting expression for position of the ship as a function of time \( s(t) \)
\[
s_f = s(t) = \frac{4}{3} \ln(6t + 1)
\]
can now be used to find the particular displacement/distance at \( t=1/6 \) hr!
\[
s(1/6) = \frac{4}{3} \ln(6(1/6) + 1) = \frac{4}{3} \ln(2) \text{ (nautical miles)}
\]
Rectilinear Kinematics: Accel. a function of velocity – a(v)

(B) METHOD 2: With \( a, v \) & \( s \) parameters given/requested, use \( ads = vdv \) form

\[
s_f = s(v) = \int_v^{v_0} \frac{v}{a(v)} \, dv + s_i \quad \Rightarrow \quad s(4) = \int_8^4 \frac{vdv}{-3/4v^2} + 0
\]

and the boat’s displacement (position?) can again be found by integration

\[
s(4) = \frac{-4}{3} \int_8^4 \frac{dv}{v} = -\frac{4}{3} \ln v \bigg|_8^4 = -\frac{4}{3} \left( \ln 4 - \ln 8 \right) = \frac{4}{3} \ln \frac{8}{4}
\]

and as was seen before

\[
s(t = 1/6) \Rightarrow s(v = 4) = \frac{4}{3} \ln(2) \quad \text{(nautical miles)}
\]

Q.E.D.

Curvilinear Kinematics Summary:

- Position
  \[
  \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}
  = r(t)\mathbf{e}_r + z\mathbf{e}_z
  \]

- Velocity: \( \mathbf{v} = \frac{d}{dt} \mathbf{r} \)
  \[
  \mathbf{v}(t) = \mathbf{r}(t) = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}
  \]
  \[
  = v\mathbf{e}_r = s\mathbf{e}_r + r\theta\mathbf{e}_\theta + z\mathbf{e}_z
  \]

- Acceleration: \( \mathbf{a} = \frac{d}{dt} \mathbf{v} \)
  \[
  \mathbf{a}(t) = \mathbf{v}(t) = \mathbf{r}(t) = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}
  = v\mathbf{e}_r + \frac{v^2}{\rho}\mathbf{e}_n
  = \left( \dot{r} - r\theta \right)\mathbf{e}_r + \left( \dot{\theta} + 2r\dot{\theta} \right)\mathbf{e}_\theta + z\mathbf{e}_z
  \]
Curvilinear Kinematics Summary:

- **Position**
  \[ \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} = \mathbf{r}(t)\mathbf{e}_r + \mathbf{z}\mathbf{e}_z \]

- **Velocity:**
  \[ \mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} = \mathbf{v}_e = \rho\mathbf{e}_e + r\theta\mathbf{e}_\theta + z\mathbf{e}_z \]

- **Acceleration:**
  \[ \mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} = \mathbf{a}_e = \mathbf{v}_e^2\mathbf{e}_e + \frac{\rho^2}{\rho}\mathbf{e}_n = \left(\rho - r\theta\right)\mathbf{e}_e + \left(r\theta + 2r\theta\right)\mathbf{e}_\theta + z\mathbf{e}_z \]

2D Curvilinear Motion: Coordinates & Conversions

- **Cartesian <-> Polar <-> Path**
  \[ \mathbf{r}(t) = x\mathbf{i} + y\mathbf{j} = r\mathbf{e}_r, \quad \mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \quad \mathbf{e}_\theta = k \times \mathbf{e}_r = \cos\theta\mathbf{j} - \sin\theta\mathbf{i}, \quad \mathbf{i} = \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta, \quad \mathbf{j} = k \times \mathbf{i} = \cos\theta\mathbf{e}_n + \sin\theta\mathbf{e}_r \]

  \[ \mathbf{v}(t) = \dot{\mathbf{r}}(t) = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} = \mathbf{v}_e = s\mathbf{e}_s, \quad \mathbf{e}_s = \frac{\mathbf{v}}{v} = \frac{x}{v}\mathbf{i} + \frac{y}{v}\mathbf{j}, \quad \mathbf{e}_n = k \times \mathbf{e}_s = \frac{x}{v}\mathbf{j} - \frac{y}{v}\mathbf{i} \]

  \[ \mathbf{i} = \cos\Psi\mathbf{e}_s - \sin\Psi\mathbf{e}_n, \quad \mathbf{j} = k \times \mathbf{i} = \cos\Psi\mathbf{e}_n + \sin\Psi\mathbf{e}_s \]
Curvilinear Motion: Cartesian Coordinates

- Projectile Motion
  - Scale w.r.t. earth such that gravity $\mathbf{g}$ is ~constant
    
    \[ |\mathbf{g}| = 32.2 \text{ ft/s}^2 = 9.81 \text{ m/s}^2 \]
    
  - Neglect any air resistance
  - Motion is PARABOLIC thus PLANAR!
  - Typically align
    - $y$-axis along gravity vector
    - $x$-axis horizontal in direction of motion
    
    \[ \mathbf{a}(t) = 0 \hat{i} - g \hat{j} = [0, -g] \]
    
    - $z$ component drops out!

Curvilinear Motion: Projectile Motion

- Integrate rectilinear relations
  - Two (2) scalar relations
  - One VECTOR relationship

\[
\begin{align*}
\mathbf{a}(t) &= 0 \hat{i} - g \hat{j} = [0, -g] = \mathbf{a}_c \\
\Rightarrow \int_{t_i}^{t_f} d\mathbf{v} &= \int_{t_i}^{t_f} \mathbf{a}_c \, dt \\
\Rightarrow \mathbf{v}_f &= \mathbf{v}_i + \int_{t_i}^{t_f} \mathbf{a}_c \, dt \\
\Rightarrow \mathbf{v}_f &= a_x (t_f - t_i) + \mathbf{v}_i \\
\Rightarrow \mathbf{r}_f &= \int_{t_i}^{t_f} \mathbf{r} = \int_{t_i}^{t_f} \mathbf{v}(t) \, dt \\
= \frac{a_x}{2} (t_f - t_i)^2 + \mathbf{v}_i (t_f - t_i) + \mathbf{r}_i \\
x_f &= v_{x_i} (t_f - t_i) + x_i \\
y_f &= \frac{-g}{2} (t_f - t_i)^2 + v_{y_i} (t_f - t_i) + y_i
\end{align*}
\]
Curvilinear Kinematics: Projectile Motion example

Given:
- Figure shown w/ ground $y = -kx^2$
- $t_0=0$, $(x_0,y_0)=0$, $v_0=v_0 \theta$ above horizon

Find: In terms of $v\theta$, $\theta$ & $k$
- (A) The location at impact $(x_I,y_I)$
- (B) Velocity & Speed @ impact, $v_I$, $v_I$
- (C) Elapsed time @ impact, $t_I$

Solution:
- 2D projectile motion
- Get expressions for $v_x(t), v_y(t)$ then $x(t), y(t)$
- Substitute into ground constraint expression
  - Solve for time of impact
- With $t_I$ known, substitute & solve for $(x_I,y_I)$

\[
\begin{align*}
\Rightarrow & \quad \mathbf{v}_f(t) = \mathbf{a}_c(t_f-t_i) + \mathbf{v}_i \\
\Rightarrow & \quad \mathbf{r}_f = \frac{\mathbf{a}_c}{2} (t_f-t_i)^2 + \mathbf{v}_i (t_f-t_i) + \mathbf{r}_i
\end{align*}
\]

Curvilinear Kinematics: Projectile Motion

- IC's $\Rightarrow t_0=0$, $(x_0,y_0)=0$, $v_0=v_0 \theta$
  $\mathbf{a}(t) = 0 \mathbf{i} - g \mathbf{j} = [0, -g]

\Rightarrow (B) \quad \mathbf{v}_f = \mathbf{a}(t_f-0) + \mathbf{v}_0 = \begin{bmatrix} v_{x_f}, v_{y_f} \end{bmatrix}

\begin{align*}
v_{x_f} &= v_0 \cos \theta \\
v_{y_f} &= -gt_f + v_0 \sin \theta
\end{align*}

- Speed
  $\begin{align*}
  s &= v = \sqrt{v_{x_f}^2 + v_{y_f}^2} \\
  &= \sqrt{(v_0 \cos \theta)^2 + (-gt_f + v_0 \sin \theta)^2} \\
  &= \sqrt{v_0^2 - 2gv_0 \sin \theta t_f + (gt_f)^2}
\end{align*}$
Curvilinear Kinematics: Projectile Motion

\[ \Rightarrow (B) \quad \mathbf{v}_t = \mathbf{a}(t_f - t_i) + \mathbf{v}_i = \begin{bmatrix} v_{x_1} \\ v_{y_1} \end{bmatrix} \]

\[ \Rightarrow (A) \quad \mathbf{r}_t = \frac{\mathbf{a}}{2} (t_f - t_i)^2 + \mathbf{v}_0 (t_f - t_i) + \mathbf{r}_i \]

\[
\begin{align*}
    x_t &= v_0 \cos \theta \ t_t \\
    y_t &= \frac{-g}{2} t_t^2 + v_0 \sin \theta \ t_t \\
    y &= -k x^2 \\
\end{align*}
\]

\[ \Rightarrow (C) \quad t_t = \frac{2v_0 \sin \theta}{g - 2k (v_0 \cos \theta)^2}, \quad t_f = 0 \]

Substitute value for \( t_f \) into position, velocity & speed relations for solution

Curvilinear Motion: Projectile Motion

– Other typical P.M. queries
  - Max Height
  - Max Range
  - Time @ some place along trajectory
  - Later w/ Path & Polar Coord
    - Velocity (speed,direction/tangent)
    - Curvature, rate of speed change ….

\[ \mathbf{a}(t) = 0 \mathbf{i} - g \mathbf{j} = [0, -g] \]

\[ \Rightarrow \mathbf{v}_f = \mathbf{a}(t_f - t_i) + \mathbf{v}_i \quad \Rightarrow \mathbf{r}_f = \frac{\mathbf{a}}{2} (t_f - t_i)^2 + \mathbf{v}_i (t_f - t_i) + \mathbf{r}_i \]

– Reconsider problems w/ different axes placement
Given: launch at 3600 m altitude \( v_o = 180 \) m/s angle 30°

\[
\begin{align*}
\dot{x} &= 0 \\
\ddot{x} &= 180 (\cos 30) = 156 \\
x &= 156 \ t \\
\text{for } h \text{ set } \dot{y} &= 0 \quad t &= 9.17 \quad h = y = 4013 \text{ m} \\
\text{for } t \text{ set } y &= 0 \quad t &= 31.18
\end{align*}
\]

Path Coord. Example ref: Meriam\&Kraige 2-8

Given:
- A rocket at high altitude with
- \( a_o = 6i - 9j \) (m/s²)
- \( v_o = 20 \) (km/hr) @ 15° below horizontal

Find: At instant given

(A) The **normal & tangential** accelerations
(B) Rate at which *speed* is increasing
(C) **Radius of curvature** of the path
(D) Angular **rotation rate** of the radial from CG to center of curvature

Solution:
- “High altitude” means negligible air resistance
- Interested only at this instant (NO Integration required)
- \( v \) given is TANGENT TO THE PATH
  - Use this to relate path to cartesian coordinates
Path Coord. Example ref: Meriam&Kraige 2-8

Solution (cont’d):

\[ \mathbf{e}_e = \frac{\mathbf{v}}{v} = \frac{\mathbf{v}}{v} = \cos 15^\circ \mathbf{i} - \sin 15^\circ \mathbf{j} \]

\[ \mathbf{e}_a = \pm (\mathbf{k} \times \mathbf{e}_e) \quad (2D \text{ shortcut!}) \]

\[ = -\cos 15^\circ \mathbf{j} - \sin 15^\circ \mathbf{i} \]

(A) \( \mathbf{a}_n \) & \( \mathbf{a}_i = \)?

\[ |\mathbf{a}_i| = \mathbf{a} \cdot \mathbf{e}_e = (6 \mathbf{i} - 9 \mathbf{j}) \cdot (\cos 15^\circ \mathbf{i} - \sin 15^\circ \mathbf{j}) = 8.12 \text{ (m/s}^2) = \mathbf{a}_i \]

\[ |\mathbf{a}_n| = \mathbf{a} \cdot \mathbf{e}_n = (6 \mathbf{i} - 9 \mathbf{j}) \cdot (-\cos 15^\circ \mathbf{j} - \sin 15^\circ \mathbf{i}) = 7.14 \text{ (m/s}^2) = \mathbf{a}_n \]

(B) \( \mathbf{v} = ?? \quad \mathbf{v} = |\mathbf{v}| = 8.12 \text{ (m/s}^2) \)

(C) \( \rho = \frac{\mathbf{v}^2}{\mathbf{a}_n} \quad \Rightarrow \quad \rho = \frac{\mathbf{v}^2}{\mathbf{a}_n} = \frac{20 \text{ km/hr}^2}{7.14 \text{ (m/s}^2)} \left( \frac{1 \text{ hr}}{3600 \text{ s}} \cdot \frac{10^3 \text{ m}}{1 \text{ km}} \right)^2 = 4.32 \times 10^6 \text{m} \)

Path Coord. Example ref: Meriam&Kraige 2-8

Solution (cont’d):

(D) \( \dot{\theta} = ?? \)

- Look either at \( \mathbf{a}_n \) or velocity

\[ a_n = \rho \dot{\theta}^2 \]

\[ \Rightarrow \dot{\theta} = \sqrt{\frac{a_n}{\rho}} = \sqrt{\frac{7.14 \text{ (m/s}^2)}{4.32 \times 10^6 \text{(m)}}} = 12.9 \times 10^{-4} \text{ rad/s} \]

\[ \mathbf{v} = \rho \dot{\theta} \]

\[ \Rightarrow \dot{\theta} = \frac{\mathbf{v}}{\rho} = \frac{20 \text{ km/hr}}{4.32 \times 10^6 \text{(m)}} \left( \frac{1 \text{ hr}}{3600 \text{ s}} \cdot \frac{10^3 \text{ m}}{1 \text{ km}} \right) = 12.9 \times 10^{-4} \text{rad/s} \]
**Relative Motion**

\[ \mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \]
\[ \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \]
\[ \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \]

**Special Case: Rigid Bodies**

When A & B are two points on the same rigid body then:

\[ \mathbf{v}_B = \mathbf{v}_A + \omega_{AB} \mathbf{k} \times \mathbf{AB} \mathbf{u}_{B/A} \] (2D)

- i.e. the relative motion is circular
- \( \mathbf{v}_{B/A} \) is perpendicular (\( \perp \)) to \( \mathbf{r}_{B/A} \) &
- \( |\mathbf{v}_{B/A}| = |\omega_{AB} \mathbf{AB}| \)

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**Relative Motion:** ref~DWY FE Review

**Given:**
- A river flows south at 5 m/s
- A boat can travel at 10 m/s relative to the water.

**Find:**
- In what direction should the boat head from A, in order to reach point B, directly across the river?

**Solution:**
- Set up Cartesian Axes @A
- Write Relative Velocity expression
  \[ \mathbf{v}_B = \mathbf{v}_R + \mathbf{v}_{B/R} \]
- Draw Velocity Polygon diagram
  - \( \mathbf{v}_R \) magnitude & direction known
  - \( \mathbf{v}_R \) direction known, not magnitude
  - \( \mathbf{v}_{B/R} \) magnitude known, not direction

\[ \mathbf{v}_R = 5 \text{ m/s} \]
\[ \mathbf{v}_B = 10 \text{ m/s} @\theta ? \]
Relative Motion: ref–Dwy/WNW FE Review

Solution (cont’d):

\[ \mathbf{v}_B = \mathbf{v}_R + \mathbf{v}_{B/R} \]
\[ \mathbf{v}_B \hat{i} = \mathbf{v}_R \hat{i} + \mathbf{v}_{B/R} (\cos \theta \hat{i} + \sin \theta \hat{j}) \]
\[ \mathbf{v}_B \hat{i} = -5 \hat{i} + 10 (\cos \theta \hat{i} + \sin \theta \hat{j}) \text{ (m/s)} \]

- Equating components:
  \[ \hat{i} \Rightarrow \mathbf{v}_B = 10 \cos \theta \text{ (m/s)} \]
  \[ \hat{j} \Rightarrow 0 = -5 + 10 \sin \theta \text{ (m/s)} \]

- 2 equations \( \leftrightarrow \) two unknowns
  \[ \sin \theta = \frac{5}{10} = \frac{1}{2} \text{ (m/s)} \]
  \[ \Rightarrow \theta = \sin^{-1}(\frac{1}{2}) = 30^\circ \]

- Lagniappe
  \[ \mathbf{v}_B = 10 \cos 30^\circ \text{ (m/s)} \]

Relative Motion: ref–Dwy/WNW FE Review

Alternate Solution (cont’d):

- With velocity polygon drawn, use the Law of Sines

\[ \frac{\sin \theta}{|\mathbf{v}_R|} = \frac{\sin 90^\circ}{|\mathbf{v}_{B/R}|} \]
\[ \theta = \sin^{-1}\left(\frac{|\mathbf{v}_R|}{|\mathbf{v}_{B/R}|}\frac{\sin 90^\circ}{5}\right) \]
\[ = \sin^{-1}\left(\frac{5}{10}\right) = 30^\circ \]
Relative Motion: ref—Meriam & Kraige 2/13

Given:
- Two cars \( A \) & \( B \) at the instant shown
  \[ v_A = 72 \text{ km/hr} \]
  \[ a_A = 1.2 \text{ m/s}^2 \]
  \[ v_B = 54 \text{ km/hr}, \text{ constant speed} \]

Find:
(A) \( v_{B/A} = ? \)
(B) \( a_{B/A} = ? \)

Solution:
- Convert to consistent units
  \[ (\text{km/hr})^* \frac{1}{3.6} = \text{(m/s)} \Rightarrow \]
  \[ v_A = 72(\text{km/hr}) = 20(\text{m/s}) \]
  \[ v_B = 54(\text{km/hr}) = 15(\text{m/s}) \]
- Motion RELATIVE TO \( A \) of interest
- Two coordinate axes are used
  - Simplifies \( \mathbf{v} \) & \( \mathbf{a} \) definitions
  - Illustrates "coordinate conversion" for expressing answers "in terms of" a unified set.

Relative Motion: ref—Meriam & Kraige 2/13

(A) Relative Velocity
\[
\mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A
= 15\mathbf{e}_t - 20 \mathbf{i} \text{ (m/s)}
= 15(\sin 30^\circ \mathbf{j} + \cos 30^\circ \mathbf{i}) - 20 \mathbf{i} \text{ (m/s)}
\]
\[
\mathbf{v}_{B/A} = -12.5\mathbf{j} + 13.0\mathbf{j} \text{ (m/s)} = 18 \text{ (m/s)} \angle -46^\circ
\]
- Velocity Polygon Approach (Graphical)
\[
\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}
\Rightarrow \mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A
\]
\[
\mathbf{v}_{B/A} = 18 \text{ m/s} \]
\[
\mathbf{v}_B = 15 \text{ m/s} \]

15
Relative Motion: ref—Meriam & Kraige 2/13

(B) Relative Acceleration
\[ a_A = 1.2 \mathbf{i} \text{ (m/ s}^2) \]
\[ a_B = v \mathbf{e}_v + \frac{v^2}{\rho} \mathbf{e}_n \text{ (m/ s}^2) = \frac{(15 \text{ m/ s})^2}{150 \text{ m}} \mathbf{e}_n \]
\[ a_B = 1.5 \mathbf{e}_n \text{ (m/ s}^2) \]
\[ a_{B/A} = a_B - a_A \]
\[ = 1.5 \mathbf{e}_n - 1.2 \mathbf{i} \text{ (m/ s}^2) \]
\[ = 1.5(\cos 30^\circ \mathbf{i} - \sin 30^\circ \mathbf{j}) - 1.2 \mathbf{i} \text{ (m/ s}^2) \]
\[ a_{B/A} = 0.1 \mathbf{j} - 0.75 \mathbf{j} \text{ (m/ s}^2) = 0.76(\text{m/ s}^2) @ -82^\circ \]

- Acceleration Polygon (Graphical)
\[ a_B = a_A + a_{B/A} \]
\[ \Rightarrow a_{B/A} = a_B - a_A \]

Given: A balloon at an altitude of 60 m is rising at steady rate of 4.5 m/s. A car passes below at constant speed of 72 kph.

Find: Relative rate of separation 1 second later:

\[ a_c = 0 \quad \mathbf{a}_b = 0 \text{ (m/ s}^2) \]
\[ \mathbf{v}_c = 20 \mathbf{i} \quad \mathbf{v}_b = 4.5 \mathbf{j} \text{ (m/s) \]
\[ \mathbf{r}_c = 20t \mathbf{i} \quad \mathbf{r}_b = (60 + 4.5t) \mathbf{j} \text{ (m) \]
\[ r_{B/C} = \begin{vmatrix} \mathbf{r}_b - \mathbf{r}_c \end{vmatrix} = \sqrt{\mathbf{r}_b - \mathbf{r}_c \cdot \mathbf{r}_b - \mathbf{r}_c} \]
\[ r_{B/C}^2 = (20t)^2 + (60 + 4.5t)^2 \]
\[ 2r_{B/C} r_{B/C} = 2(20t)(20) + 2(60 + 4.5t)4.5 \]
Divide through by 2 \( r_{B/C} \) & set \( t = 1 \)
\[ r_{B/C} = 690.25 / 67.52 = 10.22(\text{m/ s}) \]

Alternative Method (Vectors!):
\[ r_{B/C} = \mathbf{v}_b - \mathbf{v}_c = \mathbf{v}_b - \mathbf{v}_c \cdot \mathbf{r}_b - \mathbf{r}_c = -20.45 \cdot 67.5 \]
\[ = 690.3 / 67.5 = 10.2 \text{ (m/ s)} \]
Given: The ferris wheel rotates at $\dot{\theta} = 2 \text{ r/s}$, $\ddot{\theta} = -1 \text{ r/s}^2$ and the boy (B) walks to the right at a constant speed of 2 m/s.

Find: The velocity and acceleration of girl (G) on the ferris wheel relative to boy B

$\vec{v}_B = \frac{2}{\mathbf{i}}$ (m/s) 
$\vec{a}_B = 0$

$\vec{v}_G = 4 \text{ m/s} \cdot \vec{e}_\theta = 8 \vec{e}_\theta$ (m/s)

$\vec{a}_G = -4 \text{ m/s}^2 \cdot \vec{e}_\theta + 4 \text{ m/s}^2 \cdot \vec{e}_r$

Two points on a rigid body:

$\vec{v}_B = \vec{v}_A + \vec{v}_{BA}$

$\vec{v}_B \mathbf{i} = \vec{v}_A \mathbf{i} + \omega_{AB} \mathbf{k} \times \vec{AB} \mathbf{u}_{BA}$

$\vec{v}_B \mathbf{i} = \vec{v}_A \mathbf{i}$

$-\vec{AB} \omega_{AB}(\sin\theta \mathbf{i} + \cos\theta \mathbf{j})$

Equating $\mathbf{i}$ & $\mathbf{j}$ components:

$\mathbf{i} \to \vec{v}_A \cdot -\vec{AB} \omega_{AB} \sin\theta = 0$

$\mathbf{j} \to \vec{v}_B = \vec{AB} \omega_{AB} \cos\theta$

$\frac{\vec{v}_A}{\vec{v}_B} = \frac{\vec{AB} \omega_{AB} \sin\theta}{\vec{AB} \omega_{AB} \cos\theta}$
Using Instant Centers (IC):

\[ V_A = AC \omega_{AB} \mathbf{i} \]
\[ V_B = -BC \omega_{AB} \mathbf{j} \]

\[ AC = AB \sin \theta \]
\[ BC = AB \cos \theta \]

\[ \frac{v_A}{v_B} = \frac{AB \omega_{AB} \sin \theta}{AB \omega_{AB} \cos \theta} \]

Slider Crank Velocities Using Graphical & Instant Centers (IC):

\[ V_B = OB \omega_o = CB \omega_{AB} \]
\[ V_A = CA \omega_{AB} \]

Be sure to account for direction!

\[ V_A = (OB / CB) CA \omega_o \]
Given: \( \omega_c = 2 \text{ r/s} \)
\( \alpha_c = 6 \text{ r/s}^2 \)

Find: \( v_D, a_D \)

\( r_A = 6'' \)
\( r_B = 12'' \)
\( r_C = 8'' \)
\( \omega_C = 2 \text{ r/s} \)
\( \alpha_c = 6 \text{ r/s}^2 \)

\( V_{E1} = r_C \omega_C = 8 \times 2 = 16 \text{ in/s} \)
\( V_{E2} = 16 \times = r_B \omega_B = 12 \omega_B \)
so:\( \omega_B = 4/3 \text{ r/s} = \omega_A \)
\( V_D = V_F = \omega_A r_A = 4/3 \times 6 = 8 \text{ [in/s]} \)

\( a_{E1} = \alpha_c r_C \times = \omega_c^2 r_C \rightarrow \)
\( = 6 \times 8 + 4 \times 8 \rightarrow = 48 \times 32 \rightarrow [\text{in/s}^2] \)

\( a_{E2} = a_{E1} = 48 \times = \alpha_B r_B = \alpha_B (12) \alpha_B = 4 \text{ [in/s}^2] \)
\( a_{F} = \alpha_A r_A = 4 \times 6 = 24 \times = a_D \text{ [in/s}^2] \)
Given: \( r_o = 3' \quad r_i = 2' \quad v_o = 10 \text{ f/s} \quad \text{no slip} \)

Find: \( v_B \)

\( v_o = 10 \text{ ft/s} \quad \rightarrow \)

\( v_c = v_o + \omega r = 10 - 2 \omega = 0 \quad \rightarrow \quad v_A = ? \)

\( \omega = 5 \quad \text{or} -5 \quad \text{k} \)

\( v_B = v_o + \omega \times r_{B/o} = 10 \quad \text{i} + -5 \text{k} \times -3 \text{j} = -5 \text{i} \quad \text{[ft/s]} \quad \rightarrow \)

or \( v_B = v_c + \omega \times r_{B/c} = 0 + 5 \text{k} \times -1 \text{j} = -5 \text{i} \quad \text{[ft/s]} \quad \rightarrow \)

**Kinetics Summary**

- Three general solution approaches for establishing the governing equations of motion (EOM) => Which one to use?
  
  i) Newton’s Laws
  
  \[
  \sum \vec{F} = m \vec{a}_{CG} \quad \sum M_p = I_{CG} \alpha + r_{eff} m a_{CG}
  \]

  
  \[
  U_{A-B} = \int_{s_A}^{s_B} ma \, ds = \int_{v_A}^{v_B} m \nu \nu \, dv = \frac{1}{2} m \left( v_B^2 - v_A^2 \right) = \Delta T_{A-B}
  \]
  
  \[
  U_{NC} = \Delta T + \Delta V_g + \Delta V_e = \Delta E_{TOT}
  \]

  iii) Impulse - Momentum & Conservation of Momentum
  
  \[
  \mathbf{I} = \int \mathbf{F}_R \, dt = \int d\mathbf{L} = \Delta \mathbf{L}
  \]

  - Typical forces
  
  - Springs \( \mathbf{F} = k \left( s - s_0 \right) \)
  
  - Friction \( \mathbf{F}_f = \mu_{s/k} \mathbf{N} \)
  
  - Gravitation \( \mathbf{F} = m \mathbf{g} \)
Particle Kinetics: Free Body Diagrams

- **Free Body Diagrams:**
  - Isolate the particle/system of interest (i.e. boundaries)
  - For noting action-reaction between particles/bodies it is important to identify the **common normal-tangent @ the point of contact** (often one or the other is easily identified)

- Include ALL forces (& later => moments)
  - Field forces (gravity, electro-magnetic fields etc)
  - Viscous forces (aerodynamic drag, fluid flows, etc)
  - Contact forces (touching elements) -- Most common

- For motion over an interval --- draw in a general position!

---

**Kinetics:**

\[ \Sigma F = ma \]

\[ \Sigma M = I \alpha \]

FBD

\[ \sum M_p = I \alpha + r_{eff} ma \]
Given: the 20# force is applied to the sliding door which weighs 100 #
Find: the reactions at the frictionless roller supports

\[
\sum F_{\rightarrow} = 20 = \frac{(100)}{g} a \quad a = \frac{g}{5}
\]

\[
\sum M_{A} = 10B - 5(100) + (20) = 0 \quad B = 54 #
\]

\[
\sum M_{A} = 10B - 5(100) + (20) = 3 \left(\frac{100}{g}\right) (g/5)
\]

\[
\sum F_{\rightarrow} = A + B - 100 = 0 \quad A = 46 #
\]

Find: the reactions at the frictionless roller supports
Particle Kinetics: Path Coord Example  

Given:
• The slider \( m = 2 \text{ kg} \) fits loosely in the smooth slot of the disk which lies in a horizontal plane and rotates about a vertical axis through point \( O \).
• The slider is free to move only slightly along the slot in either direction before one (but not both) of the two wires \#1 or \#2 becomes taut.
• The disk starts from rest at time \( t = 0 \) and has a constant clockwise angular acceleration of \( \alpha = 0.5 \text{ r/s}^2 \).

Find:
(A) Determine the TENSION \( (T_2) \) in wire \#2 at \( t = 1 \text{ second} \).
(B) Determine the REACTION FORCE \( (N) \) between the slot and the block, again at \( t = 1 \text{ second} \).
(C) Determine the TIME \( (t) \) at which the tension in wire \#2 goes slack and wire \#1 becomes taut.

Solution:
• Asks for FORCES \( (T, N) \) so we must first establish kinematics (accelerations!)
• “Move only slightly” means it is effectively fixed relative to the slot/disk, thus
• The slider travels a circle about \( O \) & path \( (e_r, e_\theta) \) axes
  or polar \( (e_r, e_\theta) \) axes are convenient
Solution (continued):

- Newton’s Law can be applied along ANY two independent directions to resolve unknown reactions
  - Sum force components along \((n-t, r-\theta)\)
    \[
    T_2 \cos 45 + N \sin 45 = m \alpha r \\
    T_2 \sin 45 - N \cos 45 = -m \omega^2 r
    \]
  - OR to simplify algebra of unknowns, choose the directions along the unknown reactions and sum both forces and acceleration components
    \[
    T = m \left( \alpha r \cos 45 - \omega^2 r \sin 45 \right) = \frac{mr \sqrt{2}}{2} (\alpha - \omega^2) \\
    N = m \left( \alpha r \cos 45 + \omega^2 r \sin 45 \right) = \frac{mr \sqrt{2}}{2} (\alpha + \omega^2)
    \]
- ASIDE: This IS the geometric equivalent to simultaneously solving the first set of constraints to yield expressions for the unknowns
- Noting the similarity of the expressions \((\pm: + \text{ for } N, - \text{ for } T)\)
  \[
  N, T_2 = \frac{mr \sqrt{2}}{2} (\alpha \pm \omega^2)
  \]

Particle Kinetics: Path Coord Example

Solution (continued):

- Substituting the known expressions for \(\alpha \text{ and } \omega(t)\)
  \[
  N, T_2 = \frac{mr \sqrt{2}}{2} (\alpha \pm \omega^2)
  \]

(A) So for \(t=1\), the TENSION \(T_2\) is
  \[
  T_2 = \frac{\sqrt{2}}{20} \left( 1 - 0.5(1)^2 \right) (N) = \frac{\sqrt{2}}{40} (N) = 0.035 (N)
  \]

(B) At \(t=1\), the NORMAL REACTION \(N\) is
  \[
  N = \frac{\sqrt{2}}{20} \left( 1 + 0.5(1)^2 \right) (N) = \frac{3\sqrt{2}}{40} (N) = 0.106 (N)
  \]

(C) The time when TENSION \(T_2\) goes to zero is
  \[
  T_2 = \frac{\sqrt{2}}{20} \left( 1 - 0.5t^2 \right) (N) = 0 \Rightarrow 1 - 0.5t^2 = 0 \Rightarrow t = \sqrt{2} \Rightarrow t = 1.414 \text{ (s)}
  \]
Particle Kinetics: Path Coord Example  ref –Meriam & Kraige 3/74

**Langiappe:**
- The acceleration vector starts off completely in the lateral (i or f) direction here \((\omega=0)\). Since cables/wires/ropes cannot PUSH, only \(T_2\) can be engaged in balancing the \((r\ or \ n)\ component\) of the side wall reaction \(N\).
- The tangential acceleration component remains constant.
- As the disk speeds up \((\omega>0)\), the normal component increases.
- When the total acceleration vector aligns with the normal reaction force between the block & slot, the cord/wire tensions are both zero momentarily, and as \(T_2\) goes slack, \(T_1\) will become taut.

\[
\begin{align*}
\mathbf{a} & \bigg|_{t=0} = \mathbf{a}_f \text{ or } \mathbf{a}_g \\
\text{increasing } & \omega \\
T_1 & = T_2 = 0
\end{align*}
\]

---

**Impulse / Momentum**
\[
\int_{t_1}^{t_2} \mathbf{F} \, dt = \int_{t_1}^{t_2} m \mathbf{a} \, dt = \int_{t_1}^{t_2} m \mathbf{d} \mathbf{v} \\
= m \mathbf{v}_2 - m \mathbf{v}_1
\]

If mutual forces cancel, impulses cancel
( not true of work )

**coef. Of restitution**
\[
e = \frac{\text{rel. norm. sep. vel.}}{\text{rel. norm. app. vel.}}
\]

**Angular Momentum:**
\[
\int \mathbf{r} \times \mathbf{C} \, dt = \int \mathbf{I} \alpha \, dt \\
= \int \mathbf{I} \mathbf{d} \mathbf{\omega} \\
= \int \mathbf{I} \mathbf{\omega}_2 - \mathbf{I} \mathbf{\omega}_1
\]
Example: Conservation of Momentum

Given:
- An artillery gun ($m_G$) fires a shell ($m_p$) with a speed $v_p$

Find:
- (A) The recoil speed ($v_R$) of the gun

Solution:
- FBD of system components, just as shell leaves the gun
- Rectilinear motion (i.e. only horizontal motion of interest here)
- Propellant firing is internal to the system
  - System momentum is conserved in the horizontal direction
  \[ \Delta L_{\text{sys}} = 0 \]
  \[ \Delta L_{\text{s-system}} = m_G (v_G - 0) + m_p (v_p - 0) = 0 \]

\[ v_R = -v_G = \frac{m_p}{m_G} v_p \]

Example: Conservation of Momentum

Given:
- More often, a “muzzle velocity” ($v_{P/G}$) or speed of the shell relative to the gun barrel is specified

Find:
- (A) The recoil speed ($v_R$) of the gun

Solution:
- FBD (same), Rectilinear motion & Propellant firing is internal

\[ \Delta L_{\text{sys}} = 0 \]
\[ \Delta L_{\text{s-system}} = m_G (v_G - 0) + m_p (v_p - 0) = 0 \]

\[ v_p = v_G + v_{P/G} \]

\[ m_G v_G + m_p (v_G + v_{P/G}) = 0 \]

\[ v_R = -v_G = \left( \frac{m_p}{m_G + m_p} \right) v_{P/G} \]
Example: Conservation of Momentum

**Given:**
- Numerous examples with similar circumstances, rephrasing the wording
  - Kid(s) on a boat in still water, one jumps off
  - Car lands on a barge & skids to rest relative to barge
  - Rail cars collide & stay attached

**Find:**
- (A) The resulting speeds of each element
- (B) A time it takes to “skid to rest”

**Solution:**
- Similar conservation of momentum relations

\[ \Delta L_{sys} = 0 \]

---

Particle Kinetics: Impulse-Momentum

- **Impact Problems:**
  - Reformulation of one type of Impulse-Momentum
  - Impulsive Forces characterized by
    - LARGE MAGNITUDE
    - SHORT TIME DURATION
    - Ex: explosions, ball-bat, club-ball
  - Neglect other conventional forces of lesser effect for the short time interval
    - Springs
    - Gravity
    - Many Reaction forces (BUT NOT ALL!)
  - Good opportunity to look at the SYSTEM of particles in simplifying the problem (reactions are internal!)
Particle Kinetics: Impulse-Momentum/Impact

- Impact
  - Locate Common Normal/Tangent
    - Line of contact/impact - the NORMAL!
  - Forces of interaction
    - Equal, Opposite, Co-linear
  - Very complex internal phenomena, captured by Coefficient of Restitution
    \[ e = \left( \frac{V_{\text{relative Separation}}}{V_{\text{relative Approach}}} \right)_{\text{Normal}} \]
    (good derivation in B&J text --- READ IT!)

- Central & Oblique Impacts
  - Velocities are NOT co-linear with the line of impact (i.e. the common normal)

Particle Kinetics: Impulse-Momentum/Impact

- Solving Impact Problems!

  (1) Tangential Direction: individual particles have no net external impulsive forces in!
    \[ m_A v_{At} = m_A v_{At}^* \quad \& \quad m_B v_{Bt} = m_B v_{Bt}^* \]

  (2) System of particles: No net impulsive forces normal direction!
    \[ \Delta L_{\text{SYS}} = 0 \Rightarrow m_A v_{An} + m_B v_{Bn} = m_A v_{An}^* + m_B v_{Bn}^* \]

  (3) Coefficient of Restitution: Rel. Velocities along Common NORMAL!
    \[ e = \left( \frac{v_{\text{Relative Separation}}}{v_{\text{Relative Approach}}} \right)_{\text{Normal}} = \frac{v_{Bn} - v_{An}^*}{v_{An} - v_{Bn}} \]
    (Perfectly Plastic) \( 0 \leq e \leq 1 \) (Perfectly Elastic)
Particle Kinetics: Impulse-Momentum/Impact

- Solving constraint relations:

\[ v_{At} = v_{At}^* \text{ & } v_{Bt} = v_{Bt}^* \]

(1) \[ v_{Bn}^* = v_{Bn} + \frac{m_A}{m_B} (v_{An} - v_{An}^*) \]

(2) \[ v_{Bn}^* = e(v_{An} - v_{Bn}) + v_{An}^* \]

- From which the unknown rebound (normal) component of velocities become:

\[ v_{An}^* = \left( \frac{m_A - m_B e}{m_A + m_B} \right) v_{An} + \left( \frac{m_B}{m_A + m_B} \right) (1 + e)v_{Bn} \]

\[ v_{Bn}^* = \left( \frac{m_A}{m_A + m_B} \right) (1 + e)v_{An} + \left( \frac{m_B - m_A e}{m_A + m_B} \right) v_{Bn} \]

---

Particle Kinetics: Impulse-Momentum/Impact

- What if \( m_A \gg m_B \)?

\[ v_{At} = v_{At}^* \text{ & } v_{Bt} = v_{Bt}^* \]

(1) \[ v_{Bn}^* = v_{Bn} + \frac{m_A}{m_B} (v_{An} - v_{An}^*) \]

(2) \[ v_{Bn}^* = e(v_{An} - v_{Bn}) + v_{An}^* \]

- From which the unknown rebound (normal) component of velocities become:

\[ v_{An}^* = -e v_{An} + (1 + e)v_{Bn} \]

\[ v_{Bn}^* = v_{Bn} \]
Given: two balls of equal mass with the velocities shown collide, coefficient of rest = 0.8
Find: velocities after impact

\[ x - \text{mom.: } u_1 + u_2 = \frac{5}{\sqrt{2}} - 8 \]

Rest: \[ u_2 - u_1 = 0.8 \left[ (\frac{5}{\sqrt{2}}) + 8 \right] \]

\[ u_2 = 2.76 \quad u_1 = -7.23 \]

\[ \text{Work / Energy:} \]

\[ T_1 + V_1 + W_{NC} = T_2 + V_2 \]

where \[ T = \frac{1}{2} m \nu_c^2 = \frac{1}{2} I_c \omega^2 = \frac{1}{2} I_c \omega^2 \]

\[ V_{\text{grav}} = -mgh \]

\[ V_{\text{spring}} = \frac{1}{2} k x^2 \quad \text{where } x = l - l_o = \text{stretch} \]
Work-Energy Example  ref~Meriam & Kraige 3/11

Given:
• A crate of mass \( m \) slides down an incline
• \( m = 50 \text{ kg}, \theta = 15^\circ, \mu_k = 0.3 \)
• Reaches A with speed 4 m/s

Find:
(A) Speed of crate \( v_B \) as it reaches a point B 
10 m down the incline from A

Solution:
• Rectilinear motion, align axes accordingly - i.e. || & \( \perp \) to incline
• FBD of crate in general position (working over a motion interval here)
• No movement \( \perp \) to incline so 
\[ \sum F_y = mg \cos \theta - N = 0 \quad \Rightarrow \quad N = mg \cos \theta \]

Work-Energy Example  ref~Meriam & Kraige 3/11

Solution (cont’d):
• Work done is due to the resultant forces in direction of displacement (i.e. down incline) & includes Friction & component of Weight
\[ U_{A-B} = (mg \sin \theta - N \mu_k)\Delta x_{AB} \]
\[ = (mg \sin \theta - mg \cos \theta \mu_k)\Delta x_{AB} \]
• Principle of Work-Energy then says 
\[ U_{A-B} = \Delta T_{A-B} = T_B - T_A \]
\[ \Rightarrow \quad T_B = U_{A-B} + T_A \]
\[ \frac{1}{2}mv_B^2 = mg(\sin \theta - \cos \theta \mu_k)\Delta x_{AB} + \frac{1}{2}mv_A^2 \]
\[ v_B = \sqrt{2g(\sin \theta - \cos \theta \mu_k)\Delta x_{AB} + v_A^2} \]
\[ v_B = \sqrt{2 \times 9.81 \text{ (m/s)}^2 \times (\sin 15^\circ - \cos 15^\circ \times 0.3) \times 10 \text{ m} + (4 \text{ m/s})^2} \]
\[ v_B = 3.15 \text{ m/s} \]
Work-Energy: Example ref ~Meriam & Kraige 3/13

Given:
- Block \((m = 50 \text{ kg})\) mounted on rollers
- Massless spring w/ \(k = 80 \text{ N/m}\)
- Released from rest at \(A\) where spring has initial stretch of 0.233 m
- Cord w/ constant tension \(P = 300 \text{ N}\) attaches to block & routed over frictionless/massless (ideal) pulley @ \(C\)

Find:
1. Speed of block \(v_B\) as it reaches a point \(B\) directly under the pulley.

Solution:
- Again, rectilinear motion, align axes accordingly
- FBD of block in general position (working over a motion interval here)
- Look at alternative - include the rope in as part of the SYSTEM - reduce FBD to an ACTIVE Force Diagram!

\[
\begin{align*}
U_{AB_s} &= \int_A^B F_s dx = \int_A^B -kx dx = \frac{1}{2} kx^2 \bigg|_A^B \\
&= -\frac{1}{2} k(x_B^2 - x_A^2) \\
&= -\frac{1}{2} (80 \text{ N/m})((1.2 + 0.233)^2 - 0.233^2)(m^2)
\end{align*}
\]

Work-Energy: Example ref ~Meriam & Kraige 3/13

Solution (cont’d):

**ACTIVE Force Diagram!**

- Eliminate Normal Forces \(\perp\) to displacement @ their point of contact \{THEY DO NO WORK\}:
  - Weight (\(mg\)) & Roller reactions (\(N\))
  - Pulley force on rope (\(R\))
- Active forces DO work on the system
  - Spring Force (\(F_s\)) => opposes motion

\[
F_s = -kx
\]

\[
U_{AB_s} = -\frac{1}{2} (80 \text{ N/m})((1.2 + 0.233)^2 - 0.233^2)(m^2) = -80 \text{ Joules}\]
Work-Energy: Example  ref—Meriam & Kraige 3/13

Solution (cont’d):
- Calculate Work done on system
  - Cord Tension ($P$) $\Rightarrow$ constant
  - Displacement of $P$
    $L_{cord} = s_p + l = constant$
    $\Delta s_p = -\Delta l = l_B - l_A$
    $= \sqrt{1.2^2 + 0.9^2} - 0.9 \approx 0.61m$
    $U_{AB} = P\Delta s = 300(n) \ast 0.61(m)$
    $= 180 Joules$
- Work-Energy
  $U_{TOT} = \Delta T = T_B - T_A$
  $-80 + 180(Joules) = \frac{1}{2}mv_B^2 - 0$
  $\Rightarrow v_B = \sqrt{\frac{100(Joules) \ast 2}{50 Kg}} = 2.0 m/s$

Conservation-Energy Example  ref—Meriam & Kraige 3/17

Given:
- $m=3$ kg slider on circular track shown
- Starting from A with $v_A=0$
- $l_o=0.6$ m (unstretched), $k=350$ N/m
- $\mu=0$ (i.e. friction is negligible)

Find:
(A) Velocity of slider as it passes B

Solution:
- FBD of crate in general position (working over a motion interval here)
- Identify
  - Conservative Forces
    $mg$ (Weight/Gravity) & $F_s$ (Spring)
  - Non-working Constraint Forces
    $N$ (Track reaction force)
Conservation-Energy Example  

Solution:

- ALL Forces are either Conservative or Non-working constraints, therefore Cons. Of Energy applies!

\[ \Delta E_{TOT} = \Delta T + \Delta V_g + \Delta V_c = 0 \]

\[ \Delta T_{AB} = \frac{1}{2} m(v_B^2 - v_A^2) = \frac{1}{2} m(v_B^2 - 0) \]

\[ \Delta V_{ABg} = mg(y_B - y_A) = mg(0 - R) \]

\[ \Delta V_{ABc} = \frac{1}{2} k \left( (l_B - l_0)^2 - (l_A - l_0)^2 \right) \]

- Pulling together all components & isolating \( v_B \)

\[ v_B = \sqrt{2gR + \frac{k}{m} \left( R^2 - (\sqrt{2}R - R)^2 \right)} \]

- Incorporating numerical values of all terms

\[ v_B = \sqrt{2 \times 9.81 \text{m/s}^2 \times 0.6 \text{m} + \frac{350 \text{N/m}}{3 \text{kg}}} \left(0.6 \text{m}^2 - (\sqrt{2} \times 0.6 \text{m} - 0.6 \text{m})^2 \right) = 6.82 \text{ m/s} \]

Work/Energy  

\[ T_1 + V_1 + W_{NC} = T_2 = V_2 \]

Given:  \( k, m, \Theta \)

block released from rest with spring compressed \( \delta \)

Find:  distance travelled to stop, \( d \)

\[ T_1 = 0 \quad V_1 = \frac{1}{2} k \delta^2 \]

\[ W_{NC} = -Fd \]

where \( N = mg \cos \Theta, F = \mu_k N \) while sliding

\[ V_2 = \frac{1}{2} k (d - \delta)^2 - mgd \sin \Theta \]
Given: \( k, m, R, \Theta, I \)
released from rest, no slip, no initial deflection in spring

Find: speed after 1/2 revolution

\( T_1 = 0 \)
\( V_1 = 0 \)
\( W_{NC} = 0 \)

\[
T_2 = \frac{1}{2} m v_2^2 + \frac{1}{2} I \omega_2^2,
\]
\[
V_2 = \frac{1}{2} k (\pi R)^2 - mg \pi R \sin \Theta
\]

need also \( v_2 = R \omega_2 \)
(no slip condition)

\[
0 = \frac{1}{2} m v_2^2 + \frac{1}{2} I \left(\frac{v_2}{R}\right)^2 + \frac{1}{2} k (\pi R)^2 - mg \pi R \sin \Theta
\]

Kinetics Summary

- Three general solution approaches for establishing the governing equations of motion (EOM) => Which one to use?

  i) Newton’s Laws

\[
\sum F = m \mathbf{a}_{CG} \quad \sum M_p = I_{CG} \alpha + r_{eff} ma_{CG}
\]


\[
U_{A-B} = \int_{s_A}^{s_B} ma_{s} \, ds = \int_{v_A}^{v_B} mvdv = \frac{1}{2} m \left(v_B^2 - v_A^2\right) = \Delta T_{A-B}
\]

\[
U_{NC} = \Delta T + \Delta V_g + \Delta V_e = \Delta E_{TOT}
\]

iii) Impulse - Momentum & Conservation of Momentum

- Typical forces

\[
I = \int F_{gt} \, dt = \int dL = \Delta L
\]

- Springs \( F = k (s - s_0) \)
- Friction \( F_f = \mu s \cdot \mathbf{N} \)
- Gravitation \( \mathbf{F} = mg \)
Given: Box placed on conveyor with zero initial velocity.

Find: Time during which slip occurs

\[ V_{B} = 10 \text{ ft/s} \]
\[ \mu = 0.333 \]

\[ \Sigma F^\uparrow = N - mg = 0 \]
\[ \Sigma F^{\rightarrow} = \mu N = ma \]
\[ \mu mg = ma \quad a = \mu g \]

\[ v = \mu gt \]

Once slip stops \( v = v_B \)

so \( t = \frac{v_B}{\mu g} \)
Given: ball of radius \( r \) released from rest on incline, no slip, \( \Theta = 30^\circ \)

Find: acceleration

\[
\Sigma F = N - mg \cos 30 = 0
\]

\[
\Sigma F_x = mg \sin 30 - F = ma
\]

\[
\Sigma M_G = rF = 2/5 mr^2 \alpha
\]

if no slip, \( a = r \alpha \) so \( mg \sin 30 - 2/5 mr \alpha = mr \alpha \)

so \( \alpha = (5/7)(g/r)(1/2) \);

\( a = (5g/7)(1/2) = 5g/14 \)

\[
v = a \ t = \frac{5gt}{14}
\]

also: \( N = \frac{\sqrt{3}}{2} \ mg \rightarrow F_{\text{max}} = \mu_s N = \frac{\sqrt{3}}{2} \mu \ mg \)

\[
F = \frac{2}{5} mr \alpha = \frac{mg}{7}
\]

\( F \leq F_{\text{max}} \) requires \( \frac{1}{7} \leq \frac{\sqrt{3} \mu}{2} \)

or \( \mu \geq \frac{2}{7\sqrt{3}} \)
Given: \( v \), \( r \)

Find: \( \Phi \); tension

\[
\begin{align*}
ma &= m r \omega^2 \\
ma &= m \frac{v^2}{r}
\end{align*}
\]

\[
\Sigma F \uparrow = T \cos \Phi - mg = 0 \\
\Sigma F \rightarrow = T \sin \Phi = m \frac{v^2}{r}
\]

or \[
\Sigma F \rightarrow = mg \sin \Phi = ma \cos \Phi
\]

so \( \tan \Phi = \frac{a}{g} = \frac{v^2}{gr} \)

---

Given: \( m_1 \) released from rest, strikes \( m_2 \)

Find: max spring compression

State ① as shown
State ②
Vibrations

\[ \sum F = -kx - cx = mx \]

Or

\[ m\ddot{x} + c\dot{x} + kx = 0 \]
\[ \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \]

general

\[ \ddot{x} + 2\zeta \omega \dot{x} + \omega^2 x = 0 \]

where \( w \) = natural freq.
\( \zeta = \) damping factor

\( \zeta < 1 \) underdamped
\( \zeta > 1 \) overdamped
\( \zeta = 1 \) critically damped
Particle Kinetics: Cartesian Example ref – Meriam & Kraige 3/5

Given:

- A collar of mass $m$ slides vertically on a shaft with kinetic coefficient of friction, $\mu_k$.
- Applied force $F$ is constant but its direction varies as $\theta = kt$, $k$ = constant
- Collar starts from rest @ $\theta = 0^\circ$

Find:

(A) Magnitude of $F$ which results in collar coming to rest at $\theta = 90^\circ$.

Solution:

- Rectilinear motion! Constrained vertically so align axes accordingly
- FBD of collar in general position

Newton’s Laws

\[
\sum F_x = F \sin \theta - N = 0 \quad \Rightarrow N = F \sin \theta
\]
\[
\sum F_y = -F \cos \theta + mg + N \mu_k = ma
\]

\[
& - \frac{F (\sin \theta \mu_k - \cos \theta)}{m} + g = a_y
\]

Particle Kinetics: Cartesian Example ref – Meriam & Kraige 3/5

Solution (Cont’d):

- Don’t know $F$ but DO know it’s constant!
- Angle-time relation ($\theta = kt$) cleans up
  - Kinematic relationship variables &
  - Proper (Pos, Vel) BC’s for integration
  - Starts: $v_{yi} = 0, \theta = 0 \Rightarrow t = 0$
  - Ends: $v_{yf} = 0, \theta = \pi/2 \Rightarrow t = \pi/(2k)$
  - Start w/ general upper limits

\[
\int_0^{v_y} dv_y = \int_0^{t} \left( \frac{F}{m} (\sin(kt) \mu_k - \cos(kt)) + g \right) dt
\]
\[
v_{yf} = \left. \left[ \frac{F}{mk} (-\cos(kt) \mu_k - \sin(kt)) + gt \right] \right|_0^t
\]
\[
v_{yf} = \frac{F}{mk} \left[ (1 - \cos(kt) \mu_k - \sin(kt)) + gt \right]
\]
Particle Kinetics: Cartesian Example

Solution (continued):

- Now using the final BC for $t=\pi/2k$, $v_y=0$

$$v_y = \frac{F}{mk} \{ [1 - \cos(kt)]\mu_k - \sin(kt) \} + gt$$

$$0 = \frac{F}{mk} \{ [1 - \cos(\pi/2)]\mu_k - \sin(\pi/2) \} + \frac{g\pi}{2k}$$

$$0 = \frac{F}{mk} \{ \mu_k - 1 \} + \frac{g\pi}{2k}$$

$$\therefore F = \frac{\pi mg}{2(1 - \mu_k)}$$

Langiappe:

- What are:
  - The collar’s vertical displacement as a function of time?
  - The collar’s total distance traveled?

Particle Kinetics: Cartesian Example

Langiappe (continued):

- Returning to the expression for $v_y=f(t)$

$$v_y = \frac{dy}{dt} = \frac{F}{mk} \{ [1 - \cos(kt)]\mu_k - \sin(kt) \} + gt$$

$$\int_0^t dy = \int_0^t \left\{ \frac{F}{mk} \{ [1 - \cos(kt)]\mu_k - \sin(kt) \} + gt \right\} dt$$

$$y_f = \left[ \frac{F}{mk} \{ [t - \sin(kt)]\mu_k + \cos(kt) \} + \frac{1}{2} gt^2 \right]_0^t$$

$$y_f = \frac{F}{mk} \{ [t - \sin(\pi/2)]\mu_k + \cos(\pi/2) \} + \frac{1}{2} gt^2$$

$$y_f = \frac{F}{mk} \{ [\frac{\pi}{2k} - \sin(\pi/2)]\mu_k + \cos(\pi/2) \} - \frac{1}{2} g \left( \frac{\pi}{2} \right)^2$$

$$y_f = \frac{F}{mk} \{ [\frac{\pi}{2k} - 1]\mu_k - 1 \} + \frac{1}{2} g \left( \frac{\pi}{2} \right)^2$$
Particle Kinetics: Path Coord Example

Given:
- A box of mass \( m \) (particle) is released from rest @ top of a smooth circular track.

Find:
(A) Normal force \( N \) as a function of position \( \theta \) along the circular track.
(B) The angular velocity \( (\omega) \) of the pulley B such that the boxes don't slide onto the conveyor belt.

Solution:
- FBD of box in general position - working over motion interval \( A-B \), then instantaneous @ B!
  - Track smooth \( \Rightarrow \mu = 0 \), no friction!
- Attach normal-tangential coordinate axes
- Newton’s Laws for general \( \theta \)
  \[
  \sum F_n = N - mg \sin \theta = ma_n \]
  \[
  \sum F_t = mg \cos \theta = ma_t
  \]
- Must use kinematics to resolve unknowns!

Solution (cont'd):
- Kinematics: Circular track -> path coord
  \[
  \sum F_n = N - mg \sin \theta = ma_n = m \frac{v^2}{R}
  \]
  \[
  \sum F_t = mg \cos \theta = ma_t = m v = m s
  \]
- Using tangential direction to resolve velocity by integrating alternate form \( \Rightarrow \int_0^v dv = \int_0^\theta gR \cos \theta d\theta \)
  \[
  a_t = s = g \cos \theta \quad s = R \theta \Rightarrow ds = R d\theta
  \]
  \[
  \int_0^v dv = \int_0^\theta gR \cos \theta d\theta
  \]
  \[
  \frac{1}{2} v^2 = gR \sin \theta \bigg|_0^\theta = gR \sin \theta \quad \Rightarrow \quad N = m \frac{v^2}{R} + mg \sin \theta
  \]
  \[
  v^2 = 2gR \sin \theta
  \]
  \[
  N = 3mg \sin \theta
  \]
Solution (cont’d):

- Belt Kinematics:
  - From previous page, the box speed as it reaches B
    \[ v = \sqrt{2gR\sin\theta} \]
    \[ = \sqrt{\frac{2g}{2}} = \sqrt{2gR} \]
  - Belt must move at the same speed to avoid slip, i.e. relative velocity between belt/box is zero
    \[ v_B = v_{belt} \]
    \[ \sqrt{2gR} = \omega r \]
    
    (B) \[ \omega = \frac{\sqrt{2gR}}{r} \]